

J. R. Kline

SOLID GEOMETRY

BY

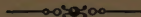
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PREFACE

THE Solid Geometry is designed as a sequel to the authors' Plane Geometry, although it can be used satisfactorily for a course in solid geometry regardless of the particular text in plane geometry which the pupils may have studied.

The Solid Geometry is shaped in accord with the same ideals that determined the form and content of the Plane Geometry. It represents an effort to incorporate, on the one hand, the spirit of the scientific treatment found in Hilbert's *Foundations of Geometry* and, on the other, the pedagogical ideals of the Report of the National Committee of Fifteen on Geometry Syllabus. (See *Mathematics Teacher*, December, 1912; *School Science and Mathematics*, 1911.)

In each Book, the fundamentally important theorems are given first. These theorems present a *safe and sane minimum course*. These are followed in each Book by one or more groups of theorems or applications which are strictly supplementary, — material which either has long appeared in geometries in some form or has been introduced in recent years for various purposes. Teachers should exercise entire freedom in omitting any of this supplementary material when conditions render such omission necessary.

An effort has been made to include suitable exercises with each of the more important theorems. In addition to those found in the main part of the text, additional exercises will be found at the close of the text.

The definitions are consistent with those given in the Plane Geometry, — all being in accord with modern scientific ideals.

Particular attention is called to the correct definition of circle as a line, and of polygon as a line. These definitions necessitate more careful phrasing of certain theorems and definitions of solid geometry in the interest of consistency; this the authors have done.

In the treatment of the mensuration of the cylinder and the cone, the fundamental limits theorems are assumed on the ground that rigorous proofs are beyond the scope of an elementary course. In the enunciation of the area theorems for portions of the surface of a sphere, changes have been made which enable pupils both to learn and to remember the theorems more readily.

The course is *practical* in the sense that the mensuration theorems for the common solids are given the place of prominence. For example, in Book IX, the mensuration of the sphere is treated in the minimum course,—the mathematically interesting theorems about spherical geometry being grouped as a supplementary topic. Besides this emphasis given to the mensuration theorems, some natural applications of solid geometry are touched upon in the exercises.

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SYMBOLS

$=$, is equal to; equals.	$\angle$, angle.
$>$, is greater than.	$\triangle$, triangle.
$<$, is less than.	$\square$, parallelogram.
$\parallel$, is parallel to; parallel.	$\square$, rectangle.
$\perp$, is perpendicular to.	$\odot$, circle.
$\perp$, perpendicular.	$\therefore$, therefore.
$\sim$, is similar to.	$\equiv$, is identically equal to.
$\cong$, is congruent to.	$\doteq$, approaches as limit.

Any symbol representing a noun is converted into the plural by affixing the letter *s*; thus $\angle$ s means angles.

ABBREVIATIONS

<i>Adj.</i> , adjacent.	<i>Ex.</i> , exercise.
<i>Alt.</i> , alternate.	<i>Ext.</i> , exterior.
<i>Ax.</i> , axiom.	<i>Hom.</i> , homologous.
<i>Con.</i> , conclusion.	<i>Hyp.</i> , hypothesis.
<i>Cong.</i> , congruent.	<i>Int.</i> , interior.
<i>Const.</i> , construction.	<i>Rect.</i> , rectangle.
<i>Cor.</i> , corollary.	<i>Rt.</i> , right.
<i>Corres.</i> , corresponding.	<i>St.</i> , straight.
<i>Def.</i> , definition.	<i>Supp.</i> , supplementary.

It is not necessary to learn any of these symbols until they are introduced in the text.

SOLID GEOMETRY

BOOK VI

LINES AND PLANES—POLYEDRAL ANGLES

442. Surfaces. No satisfactory elementary definition of surface in general can be given. The surface of a physical object is that part of the object which can, in general, be touched; it separates the portion of space occupied by the object from surrounding space.

The surface of a small pond on a calm day is approximately a *plane surface*.

The surface of a croquet ball or of a billiard ball is a *spherical surface*.

The surface of a "round" marble column is a *cylindrical surface*.

Ex. 1. If two points on the surface of a ball were joined by a straight line, where would the line lie?

Ex. 2. Are there any two points on the surface of a ball such that the straight line through them lies upon the surface of the ball?

Ex. 3. Are there two points on the surface of a cylindrical column such that the straight line joining them lies on the surface of the column?

Ex. 4. Does the straight line joining *every* pair of points on the surface of a cylindrical column lie upon the surface of the column?

443. A **Plane** is a surface such that the straight line joining any two points of it lies wholly in it.

A plane is represented to the eye by a quadrilateral like the figure adjoining.

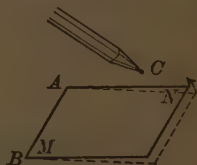
The plane may be referred to as plane $ABCD$, as plane AC , or as plane M .



Pupils will find it convenient at times to represent a plane by a thin card.

Ex. 5. In plane geometry, it was agreed that a straight line is indefinite in extent. Do you think we should agree that a plane is indefinite in extent? Why?

Ex. 6. Let MN represent a thin card of which AB is an edge. Let C represent the "point" of a pencil lying above MN . Let MN be turned about AB as an axis in the direction indicated by the arrow.



Will the card eventually come in contact with point C ?

If AB and C are kept stationary, will the card come in contact with C in more than one position of the card?

444. A plane is **determined** by a combination of points and lines if it is the only plane which contains those points and lines.

Points and lines lying in the same plane are said to be **Co-planar**.

445. Postulate. *A plane can be extended indefinitely.*

446. Axiom. *A plane is determined by three non-collinear* points.*

Ex. 7. Do two straight lines drawn at random necessarily lie in a plane? Illustrate by holding two pencils.

Ex. 8. Do four points usually lie in a plane? Select four in the school-room that do and four that do not.

Ex. 9. Are two straight lines in space which do not meet no matter how far they are extended parallel?

Ex. 10. Why is a tripod used as mounting for a camera or a surveyor's instrument?

Ex. 11. Why does a stool with three legs stand firmly whereas one with four legs cannot always be made to stand firmly?

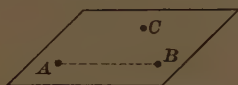
Ex. 12. Prove that a plane and a straight line not lying in the plane can have only one common point.

* Non-collinear points are points which do not all lie in one straight line.

PROPOSITION I. THEOREM

447. A plane is determined by

- I. A straight line and a point outside the line.
- II. Two intersecting straight lines.
- III. Two parallel straight lines.



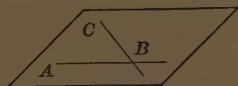
I. **Hypothesis.** Point C lies outside st. line AB .

Conclusion. C and AB determine a plane.

Proof. 1. Points A , B , and C lie in one and only one plane. § 446

2. AB lies in that plane. § 443

3. $\therefore AB$ and C determine a plane. § 444



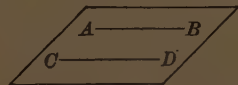
II. **Hypothesis.** AB and BC are intersecting st. lines.

Conclusion. AB and BC determine a plane.

Proof. 1. AB and point C determine a plane. § 447, I

2. BC lies in that plane. § 443

3. $\therefore AB$ and BC determine a plane.



III. **Hypothesis.** AB and CD are parallel straight lines.

Conclusion. AB and CD determine a plane.

Proof. 1. AB and CD lie in a plane. § 89

2. AB and CD cannot lie in more than one plane, for, if they did, points A , B , and C would lie in more than one plane, which is impossible. Why?

3. $\therefore AB$ and CD determine a plane.

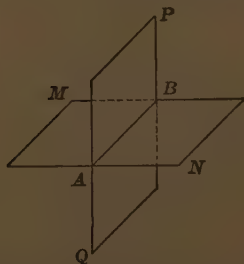
448. The **Intersection** of two surfaces or of a surface and a line consists of all points common to the surfaces, or to the surface and the line.

449. If a straight line intersects a plane, the point of intersection of the line and the plane is called the **Foot** of the line.

450. Axiom. *If two planes intersect, they have at least two common points.*

PROPOSITION II. THEOREM

451. *The intersection of two planes is a straight line.*



Hypothesis. A and B are two points common to planes MN and PQ .

Conclusion. The intersection of MN and PQ is a straight line.

Proof. 1. Draw straight line AB .

2. AB lies in plane MN and also in plane PQ . Why?

3. No point outside AB can be in both MN and PQ , for, if there were, MN and PQ would coincide. § 447, I

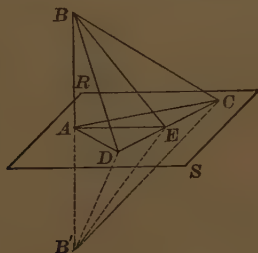
4. Hence the complete intersection of MN and PQ is the straight line AB . § 448

452. A line is **perpendicular to a plane** if it is perpendicular to every line in the plane passing through its foot.

The **plane** is also **perpendicular to the line**.

PROPOSITION III. THEOREM

453. *If a line is perpendicular to each of two intersecting lines at their intersection, it is perpendicular to their plane.*



Hypothesis. AD and AC intersect at A , determining plane RS .

$BA \perp AD$ and $BA \perp AC$.

Conclusion. $BA \perp$ plane RS .

Proof. 1. Let AE be any other straight line in RS through A . Let DC be a straight line in RS intersecting AD , AE , and AC , at D , E , and C , respectively.

2. Extend BA to B' , making $B'A = BA$.

Draw BD , BE , BC , $B'D$, $B'E$, and $B'C$.

3. AD and AC are $\perp$ bisectors of BB' . Why?

4. $\therefore BD = B'D$ and $BC = B'C$. Prove it.

5. $\therefore \triangle BDC \cong \triangle B'DC$. Prove it.

6. Revolve $\triangle B'DC$ on DC as axis until B' falls on B .

7. Then $B'E \equiv BE$. Why?

8. $\therefore A$ and E are both equidistant from B and B' .

9. $\therefore AE \perp BB'$, or $BB' \perp AE$. § 77

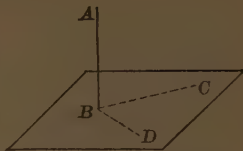
10. But AE is any st. line in RS through A .

11. $\therefore AB \perp$ every st. line in RS through A , and hence $AB \perp$ plane RS . § 452

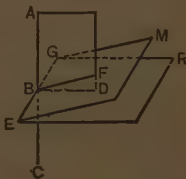
Ex. 13. If a line is perpendicular to a line of a plane, is it perpendicular to the plane?

454. Cor. 1. *Through a point of a line a plane can be drawn perpendicular to the line.*

Suggestion. — Draw BC and BD , any two perpendiculars to AB .



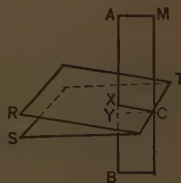
Note. — *Through a point of a line only one plane can be drawn perpendicular to the line.* If ME and RE were both $\perp$ to AB at B , a plane AD through AB would intersect ME and RE in two lines BF and BD , each $\perp$ to AB at B . But this is impossible, for, in a plane (AD), only one line can be drawn perpendicular to a given line at a point of the line.



455. Cor. 2. *Through a point outside a line, a plane can be drawn perpendicular to the line.* (See Fig. § 454.)

Suggestion. — Draw $CB \perp AB$ from C ; then draw $BD \perp AB$ at D .

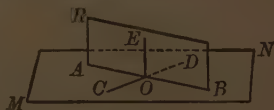
Note. — *Through a point outside a line, only one plane can be drawn perpendicular to the line.* If planes RT and ST through C were both $\perp$ to AB , the plane ABC determined by AB and C , would intersect RT and ST in lines XC and YC , each $\perp$ to AB . But this is impossible, for, in a plane, only one line can be drawn perpendicular to a given line from a point outside the line.



456. Cor. 3. *At a point in a plane, a straight line can be drawn perpendicular to the plane.*

Construction. 1. Draw CD any line in MN through O .

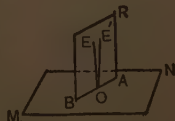
2. Draw plane $RB \perp$ to CD at O , meeting MN in line AB .



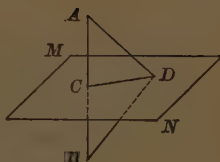
3. Draw EO in plane RB , $\perp$ to AB at O .

Statement. $EO \perp$ plane MN at O .

Note. — *At a point of a plane, only one straight line can be drawn perpendicular to the plane.* If EO and $E'O$ were both $\perp$ to plane MN at O , they would determine a plane RB which would intersect MN in line AB . EO and $E'O$ would both be $\perp$ to AB at O , and that is impossible. Why?



457. Cor. 4. Any point in the plane which is perpendicular to a segment at its mid-point is equidistant from the ends of the segment.



Ex. 14. Each of three concurrent lines is perpendicular to each of the other two. Prove that each is perpendicular to the plane of the other two.

Ex. 15. If two oblique lines, drawn to a plane from a point in a perpendicular to the plane, cut off equal distances from the foot of the perpendicular, they are equal.

Ex. 16. State and prove the converse of Ex. 15.

Ex. 17. If two oblique lines, drawn to a plane from a point in a perpendicular to the plane, cut off unequal distances from the foot of the perpendicular, the more remote is the greater.



If $BE > BD$, prove $AE > AD$.

Suggestion.—Take $BF = BD$, and draw AF .

Recall § 165

Ex. 18. State and prove the converse of Ex. 17.

Suggestion.—Give an indirect proof, basing it upon Ex. 15 and 17.

Ex. 19. If a circle be drawn in a plane and at its center a perpendicular to the plane be erected, any point in this perpendicular is equidistant from the points of the circle.

Ex. 20. A line segment of fixed length, having one extremity at a fixed point lying outside a plane, has its other extremity in the plane. What is the locus of the extremity which lies in the plane?

Ex. 21. How many different planes can be passed through one straight line?

Ex. 22. How many different planes are determined by :

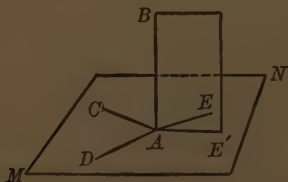
- Three concurrent lines which do not all lie in one plane?
- Three parallel lines which do not all lie in one plane?
- Two intersecting lines and a point which does not lie in their plane?
- Four points, no three of which are collinear and which do not all lie in one plane?

Ex. 23. Prove that two parallels and any transversal of them are co-planar.

Ex. 24. How many lines of intersection are determined, in general, by three planes?

PROPOSITION IV. THEOREM

458. *All the perpendiculars to a straight line at a point of the line lie in a plane perpendicular to the line at the point.*



Hypothesis. AC , AD , and AE are any three $\perp$ s to AB at A .

Conclusion. AC , AD , and AE lie in a plane $\perp$ to AB at A .

Proof. 1. Let AC and AD determine plane MN .

2. $\therefore AB \perp$ plane MN . Why?

3. Let AB and AE determine plane ABE , intersecting MN in AE' .

4. $\therefore AB \perp AE'$, since AE' is in MN . § 453

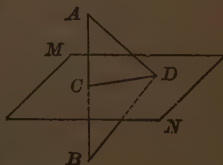
5. $\therefore AE$ and AE' , both in plane BE' , must coincide.

[In a plane, only one $\perp$ can be drawn to a line at a point in the line.] § 81

6. $\therefore AE$ must lie in MN . Why?

7. Hence all $\perp$ s to AB at A must lie in MN . Why?

459. Cor. 1. *Any point equidistant from the ends of a segment lies in the plane perpendicular to the segment at its midpoint.*

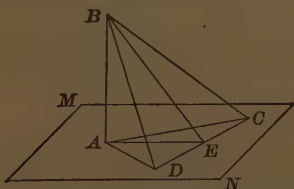


460. Cor. 2. *The locus of points in space equidistant from the ends of a segment is the plane perpendicular to the segment at its mid-point.*

Suggestion. — Review, if necessary, § 229 of the Plane Geometry and apply §§ 457 and 459.

PROPOSITION V. THEOREM

461. *If through the foot of a perpendicular to a plane a line be drawn at right angles to any line in the plane, the line drawn from its intersection with this line to any point in the perpendicular will be perpendicular to the line in the plane.*



Hypothesis. $AB \perp$ plane MN ; CD is any line in MN ; $AE \perp CD$; BE is drawn from any point B of AB to E .

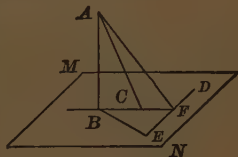
Conclusion. $BE \perp CD$.

Suggestions.—1. Take $CE = DE$, and draw BD , BC , AD , and AC .

2. Compare AC and AD .

3. Compare BD and BC .

462. Cor. 1. *From a point outside a plane, a straight line can be drawn perpendicular to the plane.*



Construction. 1. Draw DE , any st. line in MN .

2. Draw $AF \perp$ to DE at F , and BF , in MN , $\perp$ to DE at F .

3. Draw $AB \perp$ to BF .

Statement. $AB \perp$ plane MN from A .

Proof. 1. Draw BE .

2. $EF \perp$ the plane determined by AF and BF .

Why?

3. $\therefore BE \perp AB$.

§ 461

[Since BF , through the foot of EF , is $\perp$ to AB in plane ABF .]

4. $\therefore AB \perp MN$.

[See step 3 of the Construction and of the Proof.]

Note.—*From a point outside a plane only one straight line can be drawn perpendicular to the plane. If AC and AB were both $\perp$ to plane MN from A , $\triangle ABC$ would have two right angles in it.*

463. Cor. 2. *The perpendicular is the shortest segment that can be drawn from a point to a plane.*

464. The **Distance** from a point to a plane is the length of the perpendicular from the point to the plane.

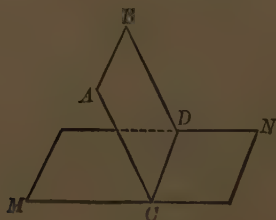
PARALLEL LINES AND PLANES

465. A straight line is **parallel to a plane** if it does not meet the plane however far they are extended.

Two **planes** are **parallel** if they do not meet however far they are extended.

PROPOSITION VI. THEOREM

466. *If a line outside a plane is parallel to a line of the plane, it is parallel to the plane.*



Hypothesis.

$AB \parallel CD.$

Plane MN contains CD but not $AB.$

Conclusion.

$AB \parallel \text{plane } MN.$

Proof. 1. AB and CD lie in a plane $AD.$

§ 89

2. This plane intersects MN in line $CD.$

Why?

3. If AB were to intersect MN , the point of intersection would be in plane MN and also in plane AD , and therefore in $CD.$

4. Hence AB would intersect $CD.$

5. But, AB cannot meet $CD.$

Why?

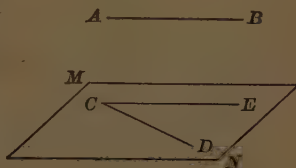
6. $\therefore AB$ cannot meet MN and hence $AB \parallel MN.$

467. Cor. 1. *Through a given straight line, a plane can be drawn parallel to any other straight line.*

Prove a plane can be drawn through $CD \parallel AB$.

Suggestion.—Draw $CE \parallel AB$.

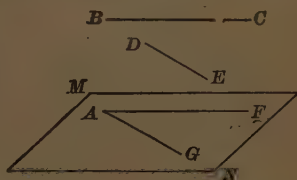
Note.—Discuss the solution when $AB \parallel CD$ and when AB is not $\parallel CD$.



468. Cor. 2. *Through a given point, a plane can be drawn parallel to each of two given straight lines in space.*

Prove a plane can be drawn through point A , parallel to straight lines BC and DE .

Suggestion.—Draw $AG \parallel DE$ and $AF \parallel BC$.



Note.—Discuss the solution when $BC \parallel DE$ and when BC is not $\parallel DE$.

Ex. 25. Can more than one perpendicular be drawn to a line at a point of the line in a plane? in space?

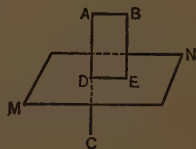
Ex. 26. Through a given point outside a line, a plane can be drawn parallel to the line. How many such planes can be drawn?

Ex. 27. Prove that a straight line and a plane, both perpendicular to the same straight line, are parallel.

Hyp. $AB \perp AC$; plane $MN \perp AC$.

Con. $AB \parallel$ plane MN .

Suggestion.—Let the plane determined by AB and AC intersect MN in line DE .

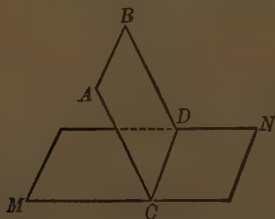


Ex. 28. Through a given point outside a plane, a line can be drawn parallel to the plane. Can more than one line be drawn parallel to the plane?

Ex. 29. Through a given point, a line can be drawn parallel to each of two intersecting planes.

PROPOSITION VII. THEOREM

469. *If a straight line is parallel to a plane, the intersection of the plane with any plane drawn through the line is parallel to the line.*



Hypothesis.

$AB \parallel \text{plane } MN.$

Plane BC , through AB , intersects MN in CD .

Conclusion.

$AB \parallel CD.$

Proof. 1. AB and CD lie in the same plane BC .

2. AB and CD cannot intersect, for if they did, AB would intersect plane MN , which is impossible.

3. $\therefore AB \parallel CD.$

§ 89

470. Cor. *If a line and a plane are parallel, a parallel to the line through any point of the plane lies in the plane.*

Hyp.

$AB \parallel \text{plane } MN.$

C is any pt. in MN .

$CD \parallel AB.$

Con.

CD lies in MN .

Proof. 1. The plane determined by AB and C intersects MN in a line CE , through C , parallel to AB .

Why?

2.

But CD , through C , $\parallel AB$.

Why?

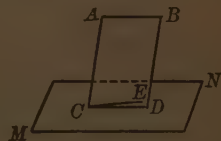
3.

$\therefore CD$ and CE coincide.

Why?

4.

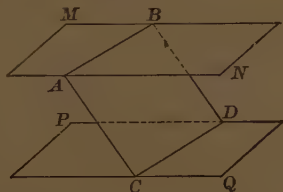
$\therefore CD$ lies in plane MN .



Ex. 30. Prove that three non-concurrent straight lines, each of which intersects the other two, lie in a plane.

PROPOSITION VIII. THEOREM

471. *If two parallel planes are cut by a third plane, the intersections are parallel.*



Hypothesis. Plane $MN \parallel$ plane PQ .

Plane AD intersects MN in AB and PQ in CD .

Conclusion. $AB \parallel CD$.

Suggestion.—Recall § 89.

Ex. 31. Prove that parallel segments between parallel planes are equal.

Ex. 32. If two planes are parallel, a line parallel to one of them through any point of the other lies in the other. (Fig. of Prop. VIII.)

Suggestion.—Given parallel planes MN and PQ , and AB through any point A of $MN \parallel PQ$. Prove that AB lies in MN . Through AB pass a plane intersecting MN in a line AB' and PQ in line CD . Consider the relation of lines AB' and CD , and also of AB and CD .

Ex. 33. From a line parallel to a plane, two parallels are drawn to the plane and terminated by the plane. Prove that the segments are equal.

Review Exercises

Ex. 34. When is a line perpendicular to a plane?

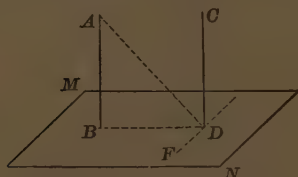
Ex. 35. Where do all lines lie which are perpendicular to a given line at a given point of the line?

Ex. 36. What is true about two lines in the same plane which are perpendicular to the same line?

Ex. 37. Line AB is perpendicular to plane MN at A . A line is drawn from B meeting any line CD of plane MN at E . If line BE is perpendicular to CD , prove AE perpendicular to CD . (Fig. Prop. V.)

PROPOSITION IX. THEOREM

472. *Two lines perpendicular to the same plane are parallel.*



Hypothesis. $AB \perp MN$ at B ; $CD \perp MN$ at D .

Conclusion. $AB \parallel CD$.

Proof. 1. Draw AD from any pt. A of AB . Draw BD .
In plane MN , draw $FD \perp BD$.

2. $CD \perp FD$. Why?

3. $AD \perp FD$. § 461

4. $\therefore AD, BD$, and CD lie in a plane $ABDC$. § 458

5. $\therefore AB$ and CD lie in the same plane, $ABDC$.

6. But $AB \perp BD$ and $CD \perp BD$. Why?

7. $\therefore AB \parallel CD$. § 97

473. Cor. 1. *If one of two parallel lines is perpendicular to a plane, the other is also perpendicular to the plane.*

Hyp. $AB \parallel CD$.
 $AB \perp$ plane MN .

Con. $CD \perp MN$.

Suggestions. — 1. Assume $CE \perp MN$.

2. Prove $CE \parallel AB$.

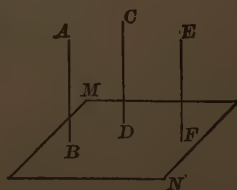


474. Cor. 2. *If each of two straight lines is parallel to a third straight line, they are parallel to each other.*

Hyp. $AB \parallel CD$; $EF \parallel CD$.

Con. $AB \parallel EF$.

Suggestion. — Let plane MN be $\perp CD$.



Ex. 38. Through a given line which is parallel to a given plane, a number of planes are passed intersecting the given plane. Prove that the lines of intersection with the given plane are all parallel.

Ex. 39. If two parallel planes intersect two other parallel planes, the four lines of intersection are parallel.

Ex. 40. Prove that a line parallel to a plane is everywhere equidistant from it.

Ex. 41. If two points are equidistant from a plane, and on the same side of it, they determine a line parallel to the plane.

Ex. 42. If two points lie on opposite sides of a plane and equidistant from it, the segment joining them is bisected by the plane.

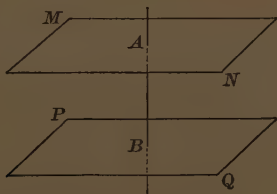
Ex. 43. If one of two parallel lines is parallel to a plane, the other is also, unless it lies in the plane.

Suggestion. — Through the line which is $\parallel$ to the plane, pass a plane intersecting the given plane. Use § 466.

Ex. 44. Prove that the lines joining in order the mid-points of the sides of a quadrilateral in space form a parallelogram.

PROPOSITION X. THEOREM

475. *Two planes perpendicular to the same straight line are parallel.*



Hypothesis. Planes MN and PQ are $\perp$ to AB .

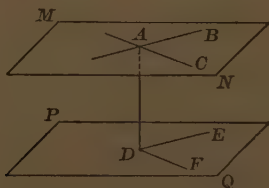
Conclusion. $MN \parallel PQ$.

Suggestion. — Prove it by the indirect method, using § 455.

Ex. 45. Are lines in space which are perpendicular to the same line necessarily parallel?

PROPOSITION XI. THEOREM

476. *If each of two intersecting lines is parallel to a plane, their plane is parallel to the given plane.*



Hypothesis. $AB \parallel \text{plane } PQ$; $AC \parallel \text{plane } PQ$
 AB and AC determine plane MN .

Conclusion. $MN \parallel PQ$.

Proof. 1. From A draw $AD \perp PQ$.

2. AC and AD determine a plane which intersects PQ in a line DF parallel to AC . Why?

3. Similarly, $DE \parallel AB$.

4. $AD \perp DF$ and also $AD \perp DE$. Why?

5. $\therefore AD \perp AC$ and also $AD \perp AB$. Why?
 Complete the proof, using § 475.

PROPOSITION XII. THEOREM

477. *A straight line perpendicular to one of two parallel planes is perpendicular to the other also.*

Hypothesis. Plane $MN \parallel \text{plane } PQ$. $AD \perp PQ$.
 (Fig. § 476.)

Conclusion. $AD \perp MN$.

Proof. 1. Through AD pass two planes intersecting MN in AB and AC , and PQ in DE and DF , respectively.

2. $\therefore AB \parallel DE$ and $AC \parallel DF$. Why?

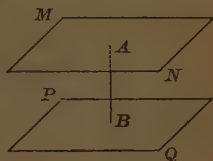
3. $AD \perp DE$ and also $AD \perp DF$. Why?

Complete the proof.

478. The distance between two parallel planes is the length of the segment perpendicular to them and lying between them.

479. Cor. 1. Two parallel planes are everywhere equally distant.

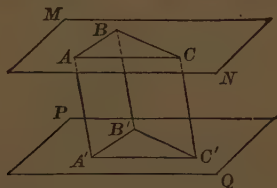
480. Cor. 2. Through a given point, a plane can be drawn parallel to a given plane.



Note. — Through a given point only one plane can be drawn parallel to a given plane. For, if two planes through A were parallel to PQ , each would be $\perp$ to AB at A , and that is impossible.

PROPOSITION XIII. THEOREM

481. If two angles not in the same plane have their sides parallel and extending in the same direction, they are equal, and their planes are parallel.



Hypothesis. $\angle BAC$ is in plane MN ; $\angle B'A'C'$ is in plane PQ .

$AB \parallel A'B'$ and extends in the same direction.

$AC \parallel A'C'$ and extends in the same direction.

Conclusion. $\angle BAC = \angle B'A'C'$; $MN \parallel PQ$.

Proof. 1. $AB \parallel$ plane PQ and $AC \parallel PQ$. § 466

2. $\therefore MN \parallel PQ$.

3. Lay off $AB = A'B'$, and $AC = A'C'$, draw BC , $B'C'$, AA' , BB' , and CC' .

4. $ABB'A'$ is a $\square$, and $\therefore BB' = AA'$ and $BB' \parallel AA'$. Why?

5. Similarly, $CC' = AA'$ and $CC' \parallel AA'$.

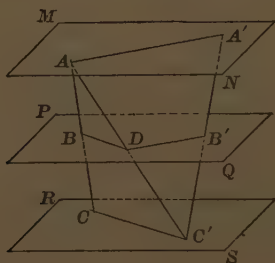
6. $\therefore BB'C'C$ is a $\square$.

Prove it

Complete the proof.

PROPOSITION XIV. THEOREM

482. *If two straight lines are cut by three or more parallel planes, the corresponding segments are proportional.*



Hypothesis. Planes MN , PQ , and RS are parallel.

AC intersects the planes at A , B , and C respectively.

$A'C'$ intersects the planes at A' , B' , and C' respectively.

Conclusion.
$$\frac{AB}{BC} = \frac{A'B'}{B'C'}.$$

Proof. 1. Draw AC' cutting PQ at D .

2. Plane CAC' intersects PQ at BD and RS at CC' .

Plane $AC'A'$ intersects PQ at DB' and MN at AA' .

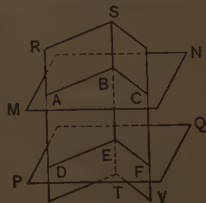
3. $\therefore BD \parallel CC'$ and also $DB' \parallel AA'$. Why?

Complete the proof, using § 265.

Ex. 46. Discuss Prop. XIII: (a) if BA and $B'A'$ extend in opposite directions and CA and $C'A'$ in the same direction; (b) if both pairs extend in opposite directions from their vertices.

Ex. 47. If each of two intersecting planes be cut by two parallel planes, not parallel to their intersection, their intersections with the parallel planes include equal angles.

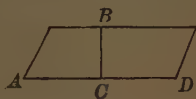
Suggestion. — Prove $\angle ABC = \angle DEF$.



Ex. 48. If two planes are parallel to a third plane, they are parallel to each other. (§ 477 and § 475.)

DIEDRAL ANGLES

483. It is evident that a straight line divides a plane into two parts, each indefinite in extent. A part of the plane, like BCD , is called a **Half-plane**. BC is called the **Edge** of the half-plane.



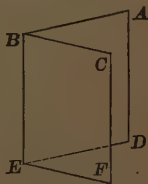
484. A **Diedral Angle** is the figure formed by two half-planes which have a common edge.

The common edge is the **Edge** and the two half-planes are the **Faces** of the diedral angle.

Thus, half-planes BF and BD form the diedral angle whose edge is BE , and whose faces are FBE and DBE .

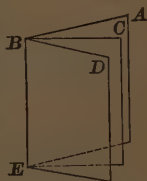
The diedral angle may be read:

diedral $\angle BE$, or diedral $\angle ABEC$.



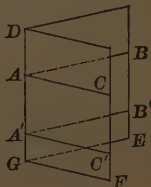
485. Two diedral angles are **Adjacent** when they have the same edge and a common face between them: as $\angle ABEC$ and $\angle CBED$.

Two diedral angles are **Vertical** when the faces of one are the extensions of the faces of the other.



486. A **Plane Angle** of a diedral angle is the angle formed by two straight lines one in each plane, drawn perpendicular to the edge at the same point.

Thus, if lines AB and AC be drawn in faces DE and DF respectively, of diedral angle DG , perpendicular to DG at A , $\angle BAC$ is a plane angle of the diedral angle DG .



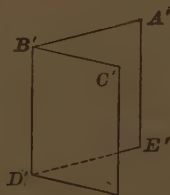
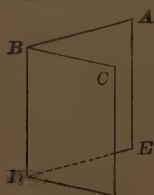
487. Cor. 1. All plane angles of a given diedral angle are equal.

488. Cor. 2. A plane perpendicular to the edge of a diedral angle intersects the faces of the angle in lines which form the plane angle of the diedral angle.

489. Two diedral angles are equal if they can be made to coincide.

PROPOSITION XV. THEOREM

490. *Two diedral angles are equal if their plane angles are equal.*



Hypothesis. $\angle ABC$ and $\angle A'B'C'$ are the plane $\angle$ s of diedral $\angle$ s BD and $B'D'$ respectively; $\angle ABC = \angle A'B'C'$.

Conclusion. $\angle BD = \angle B'D'$.

Proof. 1. Apply $\angle B'D'$ to $\angle BD$ so that $A'B'$ will coincide with AB and $B'C'$ will coincide with BC .

2. $BD \perp$ plane ABC , and $B'D' \perp$ plane $A'B'C'$. Why?

3. $\therefore B'D'$ coincides with BD . Note, § 456

4. $\therefore A'D'$ coincides with AD , and $C'D'$ with CD . § 447, II

5. $\therefore \angle B'D' = \angle BD$. § 489

491. Cor. 1. *If two diedral angles are equal, their plane angles are equal.*

For the diedral angles can be made to coincide. Then a plane, perpendicular to the common edge, will intersect in each its plane angle, and evidently these plane angles coincide.

492. Cor. 2. *If two planes intersect, the vertical diedral angles are equal.*

Suggestion. — Compare their plane angles.

Ex. 49. Prove that a plane can be drawn bisecting a diedral angle.

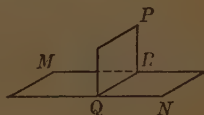
493. A diedral angle is right, acute, or obtuse according as its plane angle is right, acute, or obtuse. Two diedral angles are supplementary or complementary according as their plane angles are supplementary or complementary.

Ex. 50. If one plane meets another plane, the adjacent diedral angles formed are supplementary.

Ex. 51. If two parallel planes are cut by a third plane, the alternate-interior diedral angles are equal.

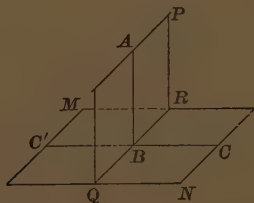
Suggestion. — Prove the plane $\angle$ s of the alt.-int. diedral $\angle$ s equal.

494. Two planes are perpendicular if the diedral angles formed are right diedral angles.



PROPOSITION XVI. THEOREM

495. If a straight line is perpendicular to a plane, every plane drawn through the line is perpendicular to the plane.



Hypothesis.

$AB \perp \text{plane } MN.$

PQ is any plane through $AB.$

Conclusion.

$PQ \perp MN.$

Proof. 1. Let PQ and MN intersect in line $QR.$

2. Draw $C'BC$ in $MN \perp QR$ at $B.$

3. $AB \perp QR.$

Why?

4. $\therefore \angle ABC$ is the plane angle of the diedral $\angle PQRN.$ Def.

5. But $\angle ABC$ is a rt. $\angle.$

Why?

Complete the proof, recalling §§ 494 and 493.

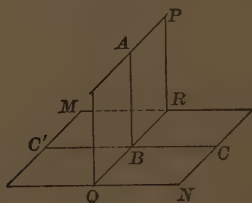
Ex. 52. (a) Prove that a plane can be drawn through a point perpendicular to a given plane.

(b) How many such planes can be drawn?

Ex. 53. Prove that a plane perpendicular to the edge of a diedral angle is perpendicular to the faces of the angle.

PROPOSITION XVII. THEOREM

496. *If two planes are perpendicular to each other, a straight line drawn in one of them perpendicular to their intersection is perpendicular to the other.*



Hypothesis. Plane $PQ \perp$ plane MN . PQ intersects MN in QR . Line AB in PQ is $\perp QR$.

Conclusion. $AB \perp MN$.

Proof. 1. Draw $C'BC$ in plane $MN \perp QR$.

2. $\therefore \angle ABC$ is the plane $\angle$ of diedral $\angle PQRN$. Why?

3. But $\angle PQRN$ is a rt. diedral $\angle$. Why?

Complete the proof.

497. Cor. 1. *If two planes are perpendicular to each other, a perpendicular to one of them at any point of their intersection lies in the other.*

Hyp. Plane $PQ \perp$ plane MN , intersecting it in QR .
 AB , drawn from any pt. B of QR , is $\perp$ to MN .

Con. AB lies in PQ .

Suggestions. — 1. A line in $PQ \perp QR$ at B is $\perp MN$.

Why?

2. Prove that it and AB coincide.

498. Cor. 2. *If two planes are perpendicular to each other, a perpendicular to one of them from any point of the other lies in the other.*

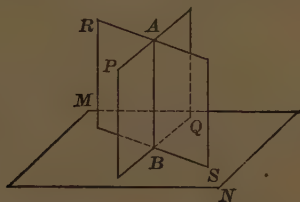
Hyp. Plane $PQ \perp$ plane MN , intersecting it in QR ;
 AB , drawn from any pt. A of PQ , is $\perp$ to MN .

Con. AB must lie in PQ .

Suggestions. — In PQ draw a $\perp$ to QR from A . Prove that AB must coincide with this perpendicular.

PROPOSITION XVIII. THEOREM

499. *A plane perpendicular to each of two intersecting planes is perpendicular to their intersection.*



Hypothesis. Plane $MN \perp$ planes RS and PQ .

RS intersects PQ in line AB .

Conclusion. $MN \perp AB$.

Suggestion. — Assume a line $\perp MN$ from A . Where will this line lie? (§ 498.)

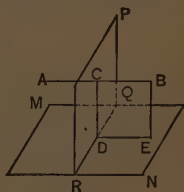
Ex. 54. Are two planes which are perpendicular to the same plane necessarily parallel?

Ex. 55. If a plane is perpendicular to a line of a plane, it is perpendicular to the plane.

Ex. 56. If a straight line is parallel to a plane, any plane perpendicular to the line is perpendicular to the plane.

Hyp. $AB \parallel$ plane MN .
Plane $PR \perp AB$ at C .

Con. $PR \perp MN$.

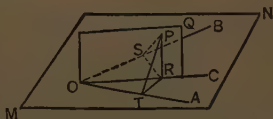


Suggestions. — 1. Draw line CD in $PR \perp QR$.

2. Let the plane determined by CB and CD intersect MN in line DE .

3. Prove $CD \perp MN$.

Ex. 57. Prove that any point in the plane through the bisector of an angle and perpendicular to the plane of the angle is equidistant from the sides of the angle.

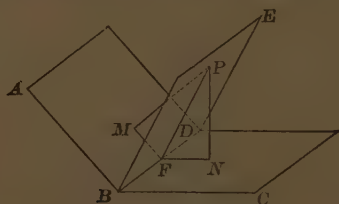


Suggestions. — 1. Draw $PR \perp OC$, $RT \perp OA$, and $RS \perp OB$. Draw PT and PS .

2. Prove $PR \perp MN$ (§ 496), $PT \perp OA$ (§ 461), $PS \perp OB$.

PROPOSITION XIX. THEOREM

500. *Every point in the plane bisecting a diedral angle is equidistant from the faces of the angle.*



Hypothesis. Plane BE bisects diedral $\angle ABDC$.

P is any point in plane BE .

$PM \perp$ plane AD ; $PN \perp$ plane DC .

Conclusion.

$PM = PN$.

Proof. 1. The plane determined by PM and PN intersects planes AD , BE , and CD in lines FM , FP , and FN respectively.

2. Plane $PMFN \perp AD$ and also $\perp CD$. Why?

3. $\therefore PMFN \perp BD$. Why?

4. $\therefore \angle PFM$ and PFN are the plane $\angle$ of diedral $\angle ABDE$ and $CBDE$. Why?

5. $\therefore \angle PFM = \angle PFN$. Why?

Complete the proof.

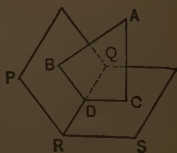
501. Cor. *Any point within a diedral angle and equidistant from its faces lies in the plane bisecting the diedral angle.*

Suggestions. — 1. Let BE be the plane determined by P and BD .

2. Prove that BE bisects $\angle ABDC$ by proving $\angle PFM$ and PFN are the plane angles of the diedral angles and are equal.

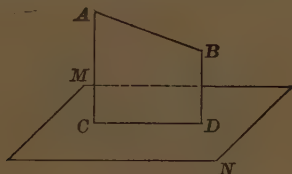
Ex. 58. If perpendiculars are drawn to the faces of a diedral angle from any point within the angle, they lie in a plane perpendicular to the edge of the diedral angle and form an angle which is the supplement of the plane angle of the diedral angle.

Suggestion. — What is the sum of the angles of a plane quadrilateral?



PROPOSITION XX. THEOREM

502. *Through a given straight line not perpendicular to a given plane, one and only one plane can be drawn perpendicular to the given plane.*



Hypothesis. AB is not $\perp$ to plane MN .

Conclusion. A plane can be drawn through $AB \perp MN$, and only one.

Proof. 1. Draw $AC \perp MN$, from point A .

2. AC and AB determine a plane $\perp MN$. Why?

3. If a second plane through AB were $\perp MN$, their intersection AB would also be $\perp MN$. Why?

4. But AB is not $\perp MN$.

5. $\therefore$ Only one plane can be drawn through $AB \perp MN$.

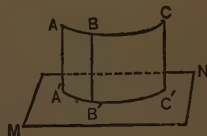
Note 1. — If AB lies in MN , the theorem is still true.

Note 2. — If $AB \perp MN$, an infinity of planes can be drawn through $AB \perp MN$ (§ 495).

503. The projection of a point on a plane is the foot of the perpendicular drawn from the point to the plane.

The projection of a given line on a plane is the line which contains the projections of all the points of the given line.

Thus, $A'B'C'$ is the projection of ABC on MN .



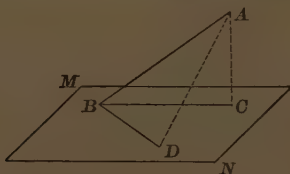
504. Cor. The projection of a straight line on a plane is a straight line. (Fig. § 502.)

Suggestions. — 1. Through AB pass a plane $AD \perp MN$.

2. Prove that the feet of the $\perp$ s to MN from AB lie in CD . (§ 498.)

PROPOSITION XXI. THEOREM

505. *The acute angle between a straight line and its projection on a plane is the least angle which it makes with any line drawn in the plane through its foot.*



Hypothesis. BC is the projection of AB on plane MN .

BD is any other line in MN through B .

Conclusion. $\angle ABC < \angle ABD$.

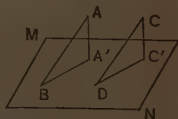
Suggestions. — 1. Take $BD = BC$. 2. Compare AD and AC .
3. Then compare $\angle ABC$ and ABD , recalling § 167.

506. The angle between a line and a plane is the acute angle made by the line with its projection on the plane. This angle is called the **Inclination** of the line to the plane.

Ex. 59. If two equal segments are drawn to a plane from a point outside the plane, they make equal angles with the plane.

Ex. 60. If two parallels meet a plane, they make equal angles with it.

Suggestion. — Given $AB \parallel CD$; $AA' \perp MN$, and $CC' \perp MN$. Prove $\angle ABA' = \angle CDC'$.



Ex. 61. Prove that a straight line and its projection upon a plane lie in a plane which is perpendicular to the given plane.

Ex. 62. If a straight line-segment is parallel to a plane, it is parallel to its projection upon the plane, and is equal to it.

Ex. 63. If two parallel lines are oblique to a plane, their projections upon the plane are parallel. (§ 481 and § 471.)

Ex. 64. Prove that the ratio of two parallel line-segments is the same as the ratio of their projections upon a given plane.

Ex. 65. Can the projection upon a plane of a curved or broken line be a straight line?

Ex. 66. If the projection upon a plane of a given figure is a straight line, then the figure lies in a plane.

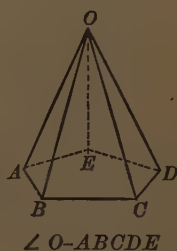
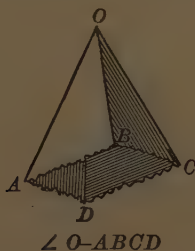
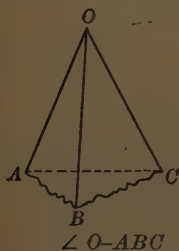
Ex. 67. Let the projection upon a given plane M of a segment l be denoted by l' . What is the relation between l and l' if:

(a) $l \perp M$? (b) $l \parallel M$? (c) l and M form an angle of 45° ?

Note. — Supplementary Exercises 1-17, p. 454, can be studied now.

POLYEDRAL ANGLES

507. The figures below represent **polyedral angles**.



Notice that each is formed of portions of three or more intersecting planes; these planes are the **Faces** of the polyedral angle. The faces intersect in one common point; this is the **Vertex** of the polyedral angle. Each face has two edges which pass through the vertex of the angle; these are the **Edges** of the polyedral angle. On each face, the two edges form an angle, whose vertex is also the vertex of the polyedral angle; these angles are the **Face Angles** of the polyedral angle. Each pair of consecutive faces intersect in an edge, forming a **diedral angle**; these diedral angles are the **Diedral Angles** of the polyedral angle.

The edges and the faces are unlimited in extent. It is convenient to indicate the polygon which results if a plane is drawn, not through the vertex, but intersecting all the faces; this polygon aids in picturing the number of faces of the polyedral angle.

508. A **Triedral Angle** is a polyedral angle having three faces.

A **Tetraedral Angle** is a polyedral angle having four faces.

Ex. 68. If a polyedral angle has 4 faces, how many vertices, edges, face angles, and diedral angles does it have?

Ex. 69. Name the edges, face angles, and the diedral angles of triedral angle $O-ABC$. (Fig. § 507.)

509. Two **polyedral angles** are **vertical** if the edges of one are the prolongations of the edges of the other.

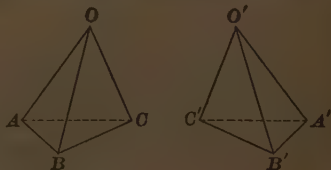
510. Two **polyedral angles** are **congruent** if they can be made to coincide.

Ex. 70. (a) Construct two triedral angles which have the face angles of one equal respectively to the face angles of the other and in the same order. Determine whether they can be made to coincide.

(b) Construct a third triedral angle whose face angles are equal to those of the triedrals of part (a), but arrange them in order opposite to that in part (a). (See Fig. § 511.) Can this triedral angle be made to coincide with either of those in part (a)?

511. Two **polyedral angles** are **symmetrical** if the face angles and the diedral angles of one are equal respectively to the face angles and the diedral angles of the other, *provided these parts occur in opposite orders*.

Thus, if face $\angle AOB$, BOC , and COA are equal respectively to face $\angle A'O'B'$, $B'O'C'$, and $C'O'A'$, and diedral $\angle O A$, $O B$, and $O C$ to diedral $\angle O' A'$, $O' B'$, and $O' C'$, triedral $\angle O-ABC$ and $O'-A'B'C'$ are symmetrical, since the parts in $\angle O'-A'B'C'$ occur in opposite order to the equal parts of $\angle O-ABC$; that is, to pass from OA to OB to OC , one moves from left to right, whereas, to pass from $O'A'$ to $O'B'$ to $O'C'$, one moves from right to left.



It is evident that, in general, two symmetrical polyedrals cannot be placed so that their faces will coincide.

PROPOSITION XXII. THEOREM

512. *Two vertical polyedral angles are symmetrical.*

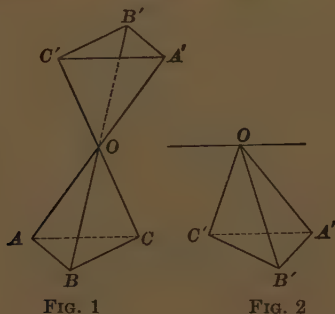


FIG. 1

FIG. 2

Hypothesis. $O-ABC$ and $O-A'B'C'$ are vertical triedral angles. (Fig. 1.)

Conclusion. $\angle O-ABC$ and $O-A'B'C'$ are symmetrical.

Proof. 1. Face $\angle AOB, BOC$, etc. equal respectively face $\angle A'OB', B'OC'$, etc. Why?

2. Dihedral $\angle BOAC$ and $B'OA'C'$ are vertical; for AOB and $A'OB'$ are parts of the same plane, as also are AOC and $A'OC'$.

Similarly, $\angle OB$ and OB' are vertical, etc.

3. $\therefore$ Dihedral $\angle OA, OB$, etc. equal respectively dihedral angles OA', OB' , etc. § 492

4. The parts of $O-ABC$ occur in opposite order to the equal parts of $\angle O-A'B'C'$.

This may be understood by moving $O-A'B'C'$ parallel to itself to the right, and then revolving it about an axis through O (as shown in Fig. 2) until face $OA'C'$ comes into the same plane as before. OB' is then in front of plane $C'OA'$ instead of back of that plane as in Fig. 1. Now, in Fig. 1, to pass from AO to OB to OC , one moves from left to right; in Fig. 2 to pass from OA' to OB' to OC' , one moves from right to left.

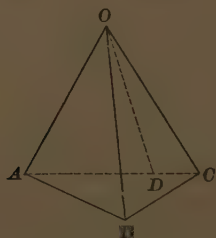
5. $\therefore \angle O-ABC$ and $O-A'B'C'$ are symmetrical. § 511

Note.—The theorem may be proved for any two vertical polyedral angles in the same manner.

PROPOSITION XXIII. THEOREM

513. *The sum of any two face angles of a triedral angle is greater than the third.*

Note. — The theorem requires proof only in the case when the third face angle is greater than either of the others.



Hypothesis.

In trihedral $\angle O-ABC$

$\angle AOC > \angle AOB$, and also $\angle AOC > \angle BOC$

Conclusion.

$\angle AOC < \angle AOB + \angle BOC$

Proof. 1. In face AOC , draw $OD = OB$, making
 $\angle AOD = \angle AOB$.

2. Through B and D pass any plane cutting the faces of the trihedral $\angle$ in AB , BC , and CA , respectively.

3. $\triangle AOB \cong \triangle AOD$.

Prove it.

4. $\therefore AB = AD$.

Why?

In $\triangle ABC$

5. $AB + BC > AD + DC$.

Why?

6. $\therefore BC > DC$.

§ 158, Ax. 18

7. $\therefore$ in $\triangle BOC$ and $\triangle COD$

$\angle BOC > \angle COD$.

§ 167

8. $\therefore \angle AOB + \angle BOC > \angle AOD + \angle COD$.

Why?

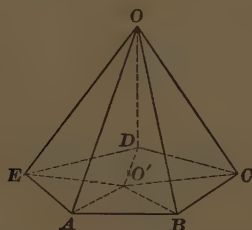
9. $\therefore \angle AOB + \angle BOC > \angle AOC$.

Ex. 71. Prove that any face angle of a polyedral angle is less than the sum of the remaining face angles.

Suggestion. — Divide the polyedral $\angle$ into trihedral $\angle$ s by passing planes through any lateral edge, and apply § 513.

PROPOSITION XXIV. THEOREM

514. *The sum of the face angles of any convex polyedral angle is less than four right angles.*



Hypothesis. $O-ABCDE$ is any convex polyedral $\angle$.

Conclusion. $\angle AOB + \angle BOC + \text{etc.} < 4 \text{ rt. } \angle$.

Proof. 1. Pass a plane cutting the faces in the polygon $ABCDE$.

Let O' be any point within $ABCDE$, and draw $O'A$, $O'B$, $O'C$, $O'D$, and $O'E$.

2. Then, in triedral $\angle A-EOB$,

$$\angle OAE + \angle OAB > \angle EAO' + \angle O'AB. \quad \S 513$$

3. Similarly $\angle OBA + \angle OBC > \angle ABO' + \angle O'BC$; etc.

4. Adding these inequalities, the sum of the base $\angle$ s of the $\triangle$ whose common vertex is O is *greater than* the sum of the base $\angle$ s of the $\triangle$ whose common vertex is O' . $\S 158$, Ax. 19

5. The sum of *all* the $\angle$ s of the $\triangle$ with vertex O equals the sum of all the $\angle$ s of the $\triangle$ with vertex O' . Why?

6. $\therefore$ the sum of the $\angle$ s at O is *less than* the sum of the $\angle$ s at O' . $\S 158$, Ax. 20

7. $\therefore$ the sum of the $\angle$ s at $O < 4 \text{ rt. } \angle$. Why?

Note. — The pupil's understanding of this theorem will be increased if a pasteboard model of the figure of $\S 514$ is at hand when this theorem is first studied.

Notice that the inequality in step 2 *does not mean* that $\angle EAO > \angle EAO'$ and $\angle OAB > \angle O'AB$; rather, the sum of $\angle EAO$ and $\angle OAB$ is greater than the face angle EAB of triedral $\angle A-EOB$.

PROPOSITION XXV. THEOREM

515. *If two trihedral angles have the face angles of one equal respectively to the face angles of the other, their homologous dihedral angles are equal.*

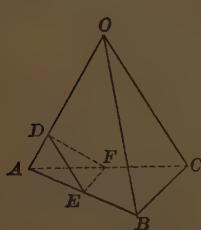


FIG. 1

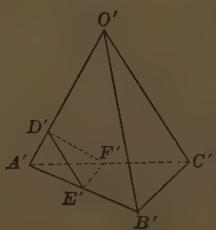


FIG. 2

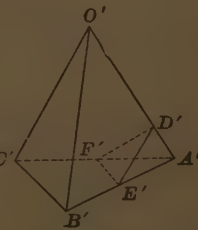


FIG. 3

Hypothesis. In trihedral $\angle O-ABC$ and $O'-A'B'C'$
 $\angle AOB = \angle A'O'B'$; $\angle BOC = \angle B'O'C'$; $\angle COA = \angle C'O'A'$.

Conclusion. Dihedral $\angle OA =$ dihedral $\angle O'A'$; etc.

Proof. 1. Lay off $OA, OB, OC, O'A', O'B',$ and $O'C'$ all of the same length, and draw $AB, BC, CA, A'B', B'C',$ and $C'A'$.

2. $\therefore \triangle OAB \cong \triangle O'A'B'$ and $AB = A'B'$. Prove it.

3. Similarly $BC = B'C'$ and $AC = A'C'$.

4. Also $\triangle ABC \cong \triangle A'B'C'$ and $\angle BAC = \angle B'A'C'$. Why?

5. On OA and $O'A'$, take $AD = A'D'$; draw DE in face $OAB \perp OA$, $D'E'$ in $A'O'B' \perp A'O'$, DF in $AOC \perp AO$, $D'F'$ in $A'O'C' \perp A'O'$, EF in face ABC , and $E'F'$ in face $A'B'C'$.

6. Then $\triangle ADE \cong \triangle A'D'E'$. Prove it.

$\therefore AE = A'E'$ and $DE = D'E'$.

7. Also $AF = A'F'$ and $DF = D'F'$. Prove it.

8. Then $\triangle AEF \cong \triangle A'E'F'$ and $EF = E'F'$. Prove it.

9. Then $\triangle DEF \cong \triangle D'E'F'$. Prove it.

10. $\therefore \angle EDF = \angle E'D'F'$. Why?

11. $\therefore$ dihedral $\angle AO =$ dihedral $\angle A'O'$. Why?

Note. — The above proof holds for Fig. 3 as well as for Fig. 2. In Figs. 1 and 2, the equal parts occur in the same order, and in Figs. 1 and 3 in the reverse order.

516. Cor. *If two triedral angles have the face angles of one equal respectively to the face angles of the other,*

1. *They are congruent if the equal parts occur in the same order.*

2. *They are symmetrical if the equal parts occur in the reverse order.*

SUPPLEMENTARY TOPICS

The following topics, theorems, and exercises of Book VI are supplementary. *Each group is independent of each of the others.* None of this material is required in the main parts of subsequent Books.

Group A. — Analogy between Triedral Angles and Triangles

Group B. — Loci in Space.

Group C. — Consists of two supplementary theorems usually given in texts.

GROUP A. ANALOGY BETWEEN TRIEDRAL ANGLES AND TRIANGLES

517. The analogy between triangles and triedral angles is very striking. Many propositions of plane geometry about triangles may be changed into propositions about triedral angles by substituting for the word *angle* of the former *diedral angle*, and for the word *side* the words *face angle*.

Ex. 72. Two triedral angles are congruent when a face angle and the adjacent diedral angles of one are equal respectively to a face angle and the adjacent diedral angles of the other, if the parts are arranged in the same order.

Suggestion. — Prove by superposition.

Note. — If the parts of one are arranged in reverse order to the parts of the other, the triedral angles are symmetrical.

Ex. 73. Two triedral angles are congruent if two face angles and the included diedral angle of one are equal respectively to two face angles and the included diedral angle of the other, if the parts are arranged in the same order.

Ex. 74. If two face angles of a triedral angle are equal, the opposite dihedral angles are equal.

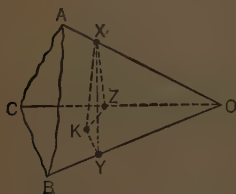
Suggestion. — Recall the proof of § 69.

Ex. 75. If two dihedral angles of a triedral angle are equal, the opposite face angles are equal.

Suggestion. — 1. Draw $XK \perp$ plane BOC ; $KY \perp BO$; $KZ \perp CO$; XY and XZ .

2. Recall § 461.

Remark. — It is assumed in this figure that K falls inside $\angle BOC$. Other possible positions of pt. K might be considered.



Ex. 76. State and prove the theorem corresponding to: *the bisector of the vertical angle of an isosceles triangle is perpendicular to and bisects the base.*

Ex. 77. If two dihedral angles of a triedral angle are unequal, the opposite face angles are unequal in the same order.

GROUP B. LOCI IN SPACE

518. The following more general definition of locus will be employed in solid geometry. The **locus of points** satisfying a given condition consists of all points which satisfy the condition, and of no other points.

The points which constitute a locus may form one (or more) lines, or one or more surfaces.

To prove that a particular assumed locus is actually the locus satisfying a given condition, or conditions, prove either (a) and (b) below or else (a) and (c).

(a) *Every point of the locus satisfies the conditions.*

(b) *Every point not of the locus does not satisfy the conditions.*

(c) *Every point which does satisfy the conditions lies in the locus.*

Ex. 78. As a consequence of § 457 and § 459, what is the locus of points equidistant from the extremities of a segment?

Ex. 79. As a consequence of § 500 and § 501, what is the locus of points equidistant from two intersecting planes?

Ex. 80. What is the locus of points in space at a given distance from a given plane?

Ex. 81. What is the locus of points in space equally distant from two parallel planes?

Ex. 82. What is the locus of points in space equidistant from the points of a given circle?

Ex. 83. What is the locus of points in space equidistant from the vertices of a given triangle?

519. Intersection Loci. — When two conditions are imposed upon a point in space, each condition determines a locus for the point, and the desired point lies in the intersection of the loci. It is often inadvisable to attempt to draw the two loci, for that demands considerable skill in drawing. The following form of solution brings out all the mathematical value of such a problem quite as well, if not better, than if a figure were drawn.

ILLUSTRATIVE PROBLEM. — *What is the locus of points in space at a given distance from a given plane and equidistant from two given points?*

Solution. 1. The locus of points at a given distance from a given plane consists of two planes parallel to the given plane and at the given distance from it. Call this Locus 1.

2. The locus of points equidistant from two given points is the plane perpendicular to and bisecting the segment between the two points. Call this Locus 2.

3. The desired locus is the intersection of Locus 1 and Locus 2.

Discussion. 1. Generally the plane of Locus 2 will intersect the planes of Locus 1 in two straight lines.

2. Locus 2 may be parallel to the planes of Locus 1. In this case, there will not be any points satisfying the given conditions.

3. Locus 2 may coincide with one of the planes of Locus 1. In this case, the plane common to Loci 1 and 2 will be the desired locus.

Ex. 84. What is the locus of points in a given plane equidistant from two parallel planes?

Ex. 85. What is the locus of points in a given plane at a given distance from another given plane?

Ex. 86. What is the locus of points in a given plane equidistant from two given points not in the plane?

Ex. 87. What is the locus of points in a given plane equidistant from two intersecting planes?

Ex. 88. What is the locus of points equidistant from two given points and also equidistant from two parallel planes?

Ex. 89. What is the locus of points equidistant from two given points and also equidistant from two intersecting planes?

Ex. 90. What is the locus of points equidistant from two parallel planes, and also equidistant from two intersecting planes?

Ex. 91. What is the locus of points equidistant from two intersecting planes, and also at a given distance from a given plane?

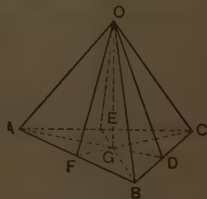
Ex. 92. Find all points which are at a given distance from a given plane, equidistant from two other parallel planes, and equidistant from two given points.

Ex. 93. Find all points which are equidistant from two given intersecting planes, equidistant from two parallel planes, and equidistant from two given points.

Ex. 94. Prove that the three planes bisecting the dihedral angles of a trihedral angle meet in a common straight line.

Suggestion.—Planes OAD and OBE intersect in line OG . Prove OG is in plane OCF .

Ex. 95. Prove that the three planes determined by the edges of a trihedral angle and the bisectors of the opposite face angles intersect in a line.



Suggestions.—1. In a figure like that for Ex. 94, assume OD , OF , and OE bisect the angles BOC , BOA , and AOC , respectively.

2. Take $OA = OB = OC$, and draw AB , BC , and AC .

3. Prove AD , CF , and EB are concurrent.

4. Prove planes AOD , BOE , and COF meet in a line.

Ex. 96. Prove that the locus of points equidistant from the sides of an angle is the plane through the bisector of the angle, and perpendicular to the plane of the angle.

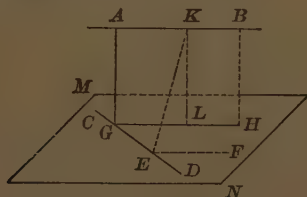
Suggestion.—Recall § 460 and Ex. 57, Book VI.

Ex. 97. Prove that the planes through the bisectors of the face angles of a trihedral angle, perpendicular to the planes of the faces, meet in a line, whose points are equidistant from the edges of the trihedral angle.

GROUP C. SUPPLEMENTARY THEOREMS

PROPOSITION XXVI. THEOREM

520. *Two straight lines not in the same plane have one common perpendicular, and only one; and this segment is the shortest segment that can be drawn between them.*



Hypothesis. AB and CD do not lie in the same plane.

Conclusion. One and only one common $\perp$ to AB and CD can be drawn; also, this $\perp$ is the shortest segment between AB and CD .

Proof. 1. Through CD draw plane $MN \parallel AB$. § 467

2. Through AB , draw plane $AH \perp$ plane MN , intersecting MN in line GH . § 502

3. $\therefore GH \parallel AB$. Why?

4. $\therefore GH$ intersects CD at a point G .

[If $CD \parallel GH$, CD would be $\parallel$ to AB , which is impossible.]

5. In plane AH , draw $AG \perp GH$ at G .

6. Then $AG \perp AB$ and also to CD . Prove it.

7. Assume KE also $\perp$ to both AB and CD .

8. Draw $EF \parallel AB$, and KL in plane $AH \perp GH$.

9. EF is in MN . § 470

10. $EK \perp EF$. Why?

11. $\therefore EK \perp MN$. Why?

12. But this is impossible for $KL \perp MN$. § 496

13. Hence AG is the only common $\perp$ to AB and CD . Why?

14. $EK > KL$. Why?

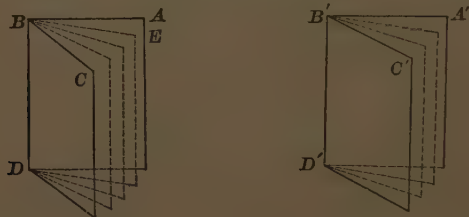
15. $\therefore EK > AG$. Why?

16. $\therefore AG$ is the shortest segment from AB to CD .

PROPOSITION XXVII. THEOREM

521. *Two diedral angles have the same ratio as their plane angles.*

CASE I. *When the plane angles are commensurable.*



Hypothesis. $\angle ABC$ and $\angle A'B'C'$, the plane $\angle$ s of diedral $\angle$ s $\angle ABDC$ and $\angle A'B'D'C'$ respectively, are commensurable.

Conclusion.
$$\frac{\angle A'B'D'C'}{\angle ABDC} = \frac{\angle A'B'C'}{\angle ABC}.$$

Proof. 1. Let $\angle ABE$, a common measure of $\angle ABC$ and $\angle A'B'C'$, be contained 4 times in $\angle ABC$ and 3 times in $\angle A'B'C'$.

2. Then
$$\frac{\angle A'B'C'}{\angle ABC} = \frac{3}{4}.$$

3. Passing planes through the edges BD and $B'D'$, and the division lines of $\angle ABC$ and $\angle A'B'C'$, respectively, diedral $\angle ABDC$ is divided into 4 parts, and $\angle A'B'D'C'$ into three parts, all of which are equal. Why?

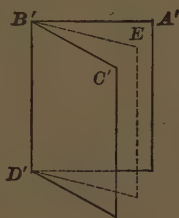
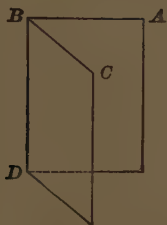
4.
$$\therefore \frac{\angle A'B'D'C'}{\angle ABDC} = \frac{3}{4}.$$

5.
$$\therefore \frac{\angle A'B'D'C'}{\angle ABDC} = \frac{\angle A'B'C'}{\angle ABC}.$$
 Why?

CASE II. *When the plane angles are incommensurable.*

Hypothesis. $\angle ABC$ and $\angle A'B'C'$, plane $\angle$ s of diedral $\angle$ s $\angle ABDC$ and $\angle A'B'D'C'$, respectively, are incommensurable.

Conclusion.
$$\frac{\angle A'B'D'C'}{\angle ABDC} = \frac{\angle A'B'C'}{\angle ABC}.$$



Proof. 1. Let $\angle ABC$ be divided into any number of equal parts, and let one of these parts be applied to $\angle A'B'C'$ as unit of measure.

Since $\angle ABC$ and $\angle A'B'C'$ are incommensurable, the unit will not be contained exactly in $\angle A'B'C'$.

A certain number of equal parts will extend from $A'B'$ to $B'E$, leaving the remainder $\angle EB'C'$ less than the unit of measure.

2. Pass a plane through $B'D'$ and $B'E$. Then

$$\frac{\angle A'B'D'E}{\angle ABDC} = \frac{\angle A'B'E}{\angle ABC}. \quad \text{Case 1}$$

3. Now let the number of subdivisions of $\angle ABC$ be indefinitely increased; then the unit of measure will be indefinitely decreased, and consequently the remainder $\angle EB'C'$ will approach the limit 0. § 401

$$4. \quad \therefore \frac{\angle A'B'D'E}{\angle ABDC} \doteq \frac{\angle A'B'D'C'}{\angle ABDC}. \quad \text{§ 403, (a)}$$

[“ $\doteq$,” means “approaches the limit.”]

$$5. \text{ Also } \frac{\angle A'B'E}{\angle ABC} \doteq \frac{\angle A'B'C'}{\angle ABC}. \quad \text{§ 403, (a)}$$

$$6. \quad \therefore \frac{\angle A'B'D'C'}{\angle ABDC} = \frac{\angle A'B'C'}{\angle ABC}. \quad \text{§ 403, (b)}$$

BOOK VII

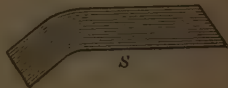
POLYEDRA

522. Surfaces. So far only *plane surfaces* have been considered.

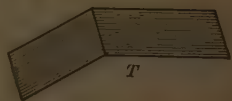
A **Curved Surface** is a surface no part of which is plane.

The surface of a spherical object is a familiar example of a curved surface.

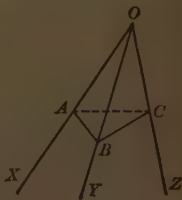
It is evident that there may be surfaces of which part is plane and part is curved; as surface *S*.



Also there are surfaces consisting of two or more parts each of which is plane; as surface *T*.



523. Closed Surfaces. Let a plane ABC intersect the faces of triedral angle $O-XYZ$, and consider the surface consisting of triangles OAB , OAC , OBC , and ABC , and the portions of planes within them. This surface incloses a *finite* portion of space. Such a surface is a *closed surface*. A closed surface is such that the intersection of it made by *every* intersecting plane is a closed line.



524. A **closed surface** is **convex** if the intersection with it of every intersecting plane is a convex closed line.

It will be assumed that all closed surfaces considered in this text are **convex**.

525. A **Solid** is the finite portion of space inclosed by a closed surface. The surface is called the *surface of the solid*, and is said to bound the solid.

In the remaining part of solid geometry a detailed study is made of certain common solids.

526. A **Polyedron** is a solid bounded by portions of planes, called the **Faces** of the polyedron. The faces intersect in straight lines, called the **Edges** of the polyedron. The edges intersect in points, called the **Vertices** of the polyedron. The straight line joining any two vertices of the polyedron which do not lie in the same face is a **Diagonal** of the polyedron.

527. The least number of planes that can form a polyedral angle is three. Then the least number of planes that can form a polyedron is four.

A polyedron of four faces is a **Tetraedron**; one of six faces is a **Hexaedron**; one of eight faces is an **Octaedron**; one of twelve faces is a **Dodecaedron**; and one of twenty is an **Icosaedron**.

The cube is a familiar hexaedron.



TETRAEDRON



PENTAEDRON



HEXAEDRON

Ex. 1. Verify for a tetraedron, a pentaedron, and a hexaedron, the formula

$$F + V = E + 2,$$

where F is the number of faces, V is the number of vertices, and E is the number of edges.

Note. — This theorem is due to the mathematician Leonard Euler.

Ex. 2. If E , F , G , and H are the mid-points of edges BD , BC , AC , and AD , respectively, of tetraedron $ABCD$, prove $EFGH$ a parallelogram.



Ex. 3. If $ABCD$ is a tetraedron, the section made by a plane parallel to each of the edges AB and CD is a parallelogram.

Note.—Remember that by § 468 a plane *can be* drawn parallel to each of two straight lines in space.

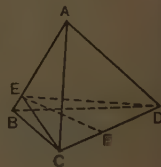
Ex. 4. The lines joining, by pairs, the mid-points of opposite edges of any tetraedron, intersect in a common point.



Ex. 5. In tetraedron $ABCD$, a plane is drawn through edge CD perpendicular to AB , intersecting faces ABC and ABD in CE and ED , respectively. If the bisector of $\angle CED$ meets CD at F , prove

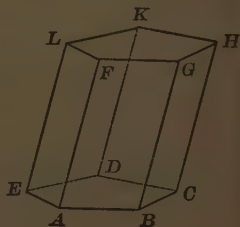
$$CF : DF = \text{area } ABC : \text{area } ABD.$$

Suggestion.—Recall § 270.



PRISMS AND PARALLELOPIPEDS

528. A **Prism** is a polyedron, two of whose faces lie in parallel planes, and whose remaining faces, in order, intersect in parallel lines. The parallel faces are the **Bases**; the other faces are the **Lateral Faces**; the edges which are not sides of the bases are the **Lateral Edges**; the perpendicular between the bases is the **Altitude**; the sum of the areas of the lateral faces is the **Lateral Area**.



If a plane is perpendicular to the lateral edges of a prism, its intersection with the prism is called a **Right Section** of the prism.

529. The following **Important Facts** about a **Prism** should be proved by the pupil:

I. *The lateral faces of a prism are inclosed by parallelograms.*

Prove $BCHG$ is a $\square$, using § 471.

Fig. § 528

II. The lateral edges of a prism are parallel and equal.

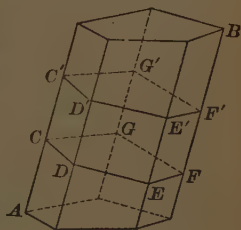
III. The bases of a prism are inclosed by congruent polygons.

Suggestion.—Recall § 481.

IV. Sections of the lateral surface of a prism made by two parallel planes cutting all the lateral edges are congruent polygons.

Suggestion.—Let planes CF and $C'F'$ be parallel.

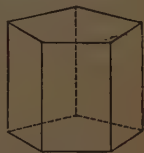
Prove $CDEFG \cong C'D'E'F'G'$.



V. A section of a prism made by a plane parallel to the base is congruent to the base.

530. Kinds of Prisms. A prism is *triangular*, *quadrangular*, etc., according as its base is *triangular*, *quadrangular*, etc.

A **Right Prism** is a prism whose lateral edges are perpendicular to its bases.



An **Oblique Prism** is a prism whose lateral edges are not perpendicular to its bases.

A **Regular Prism** is a right prism whose base is inclosed by a regular polygon.



A **Truncated Prism** is that portion of a prism bounded by the base and a plane not parallel to the base, cutting all the lateral edges.

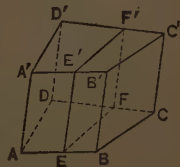
Ex. 6. Prove that every pair of lateral edges of a prism determines a plane which is parallel to each of the other lateral edges of the prism.

Ex. 7. Prove that the lateral edges of a right prism are equal to the altitude.

Ex. 8. Prove that the lateral faces of a right prism are inclosed by rectangles.

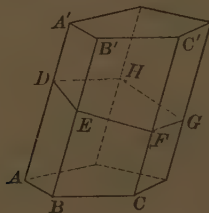
Ex. 9. Prove that the section of a prism made by a plane parallel to a lateral edge is inclosed by a parallelogram.

Suggestion.—Let plane EF' be $\parallel AA'$, a lateral edge of prism AC' .



PROPOSITION I. THEOREM

531. *The lateral area of a prism equals the perimeter of a right section multiplied by the length of a lateral edge.*



Hypothesis. $DEFGH$ is a rt. section of prism AC' .

P = perimeter of $DEFGH$; L = length of AA' ;

S = the lateral area.

Conclusion.

$$S = LP.$$

Proof. 1. Area of $\square AB' = L \times DE$.

Why?

2. Similarly, area of $\square BC' = L \times EF$.

Why?

Complete the proof.

532. Cor. *The lateral area of a right prism equals the perimeter of the base multiplied by the length of the altitude.*

Ex. 10. Find the lateral area of a regular hexagonal prism, each side of whose base is 3 and whose altitude is 9.

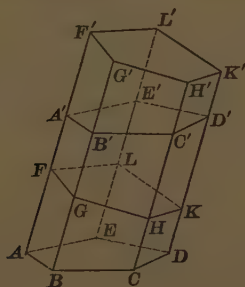
Ex. 11. There are upon a porch six columns having the form of regular octagonal prisms. If the side of the base is 4 inches and the altitude of the column is 7 feet, find the total of the lateral areas of the columns.

533. The **Volume of a Solid** is a number which indicates the measure of that solid in terms of a unit of solid measure; it is the ratio of the solid to the unit of solid measure.

534. Two **solids** are **equal** if they have equal volumes. Evidently two congruent solids are equal; likewise two solids which can be divided into parts which are respectively congruent are equal.

PROPOSITION II. THEOREM

535. *An oblique prism equals a right prism which has for its base a right section of the oblique prism, and for its altitude a lateral edge of the oblique prism.*



Hypothesis. FK' is a right prism. Its base FK is a right section of oblique prism AD' .

Its altitude $FF' = AA'$, a lateral edge of AD' .

Conclusion. $FK' = AD'$.

Proof. 1. $ABCDE \cong A'B'C'D'E'$,

and $FGHKL \cong F'G'H'K'L'$. § 529, IV

2. $AF = A'F'$; $BG = B'G'$; $CH = C'H'$; etc. Prove it.

3. Slide polyedron AK upward, letting AF , BG , etc., move along lines AF' , BG' , etc., until $ABCDE$ coincides with $A'B'C'D'E'$.

4. Then F , G , H , etc., fall upon F' , G' , H' , etc. Step 2

5. $\therefore FGHKL$ will fall upon $F'G'H'K'L'$.

6. $\therefore$ polyedron $AK \cong$ polyedron $A'K'$.

7. $\therefore$ prism $AD' =$ prism FK' . § 534

536. Cor. *Two right prisms are congruent, and hence equal, when they have congruent bases and equal altitudes.*

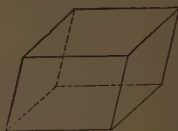
For the bases can be made to coincide. Then the lateral edges of one will coincide with the homologous edges of the other by § 530 and Note, § 456. Then the upper bases must coincide.

537. A **Parallelopiped** is a prism whose base is inclosed by a parallelogram. As a consequence, all the faces of a parallelopiped are inclosed by parallelograms.

A **Right Parallelopiped** is a parallelopiped whose lateral edges are perpendicular to its bases.

A **Rectangular Parallelopiped** is a right parallelopiped whose base is inclosed by a rectangle. Consequently all the faces are inclosed by rectangles. (See Ex. 8, p. 349.)

A **Cube** is a rectangular parallelopiped whose six faces are inclosed by squares.

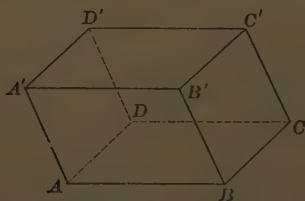


Ex. 12. How many faces of a right parallelopiped are rectangles?

Ex. 13. Is a cube a prism?

PROPOSITION III. THEOREM

538. *The opposite lateral faces of a parallelopiped are congruent and parallel.*



Hypothesis. AC and $A'C'$ are the bases of parallelopiped AC' .

Conclusion. Faces AB' and DC' are congruent and $\parallel$.

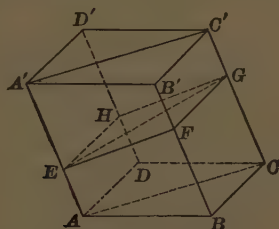
Suggestions. — 1. To prove $AB' \cong DC'$, prove them mutually equiangular and mutually equilateral.

2. To prove $AB' \parallel DC'$, recall § 481.

539. **Cor.** *Any pair of opposite faces of a parallelopiped may be taken as its bases.*

PROPOSITION IV. THEOREM

540. *The plane through two diagonally opposite edges of a parallelopiped divides it into two equal triangular prisms.*



Hypothesis. Plane AC' passes through edges AA' and CC' of parallelopiped $A'C$.

Conclusion. Prism $A'-ABC =$ prism $A'-ACD$.

Proof. 1. Let $EFGH$ be a right section of the parallelopiped, intersecting plane $AA'C'C$ in EG .

2. $EFGH$ is a $\square$. Prove it.

3. $\therefore \triangle EFG \cong \triangle EGH$. Why?

4. Prism $A'-ABC =$ a right prism with base EFG and altitude AA' .

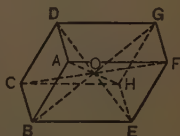
Prism $A'-ACD =$ a right prism with base EGH and altitude AA' . § 535

5. But these right prisms are equal. § 536

6. $\therefore A'-ABC = A'-ACD$.

Ex. 14. Prove that the diagonals of a rectangular parallelopiped are equal.

Ex. 15. Prove that the diagonals of a parallelopiped bisect each other.



Suggestion.—Prove that each of the other diagonals bisects BG .

Note.—The point of intersection of the diagonals of a parallelopiped is called the **center of the parallelopiped**.

Ex. 16. Prove that any line drawn through the center of a parallelopiped, terminating in a pair of opposite faces, is bisected at that point.

Ex. 17. Prove that the line joining the center of a parallelopiped to the center of any face is parallel to any edge of the parallelopiped which intersects that face, and is equal to one half of it.

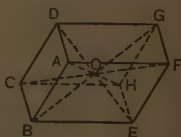
Ex. 18. Prove that the centers of two opposite faces of a parallelopiped and the center of the parallelopiped are collinear.

Suggestion.—Recall Ex. 17 and § 90.

Ex. 19. If the four diagonals of a quadrangular prism pass through a common point, the prism is a parallelopiped.

Plan.—Prove $BCHE$ is a $\square$.

Suggestion.— DE and AH determine a plane (why?) which intersects planes DF and BH in lines AD and HE , respectively. Compare AB and HE ; also compare AD and BC . Then compare BC and HE .



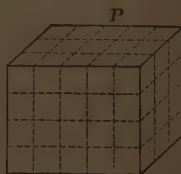
Ex. 20. Recall that the diagonals of a square are equal, bisect each other, and are perpendicular to each other.

What questions about the diagonals of a cube do these facts suggest? Prove the answers to your questions.

§ 401. The **Dimensions** of a rectangular parallelopiped are the *lengths* of its three edges which meet at any vertex.

The **Volume** of a rectangular parallelopiped is readily determined when the three dimensions are multiples of the linear unit of measure.

Thus, if the dimensions of P are 5 units, 4 units, and 3 units, respectively, the solid can be divided into 60 unit cubes. In this case 60, the number which expresses the area, is the product of 3, 4, and 5, the three dimensions.



The following three propositions prove that this is the correct formula for determining the volume of any rectangular parallelopiped. Before taking up these propositions, certain necessary new ideas are introduced.*

* The class may have studied limits in Plane Geometry (§ 401 and following). In that case the following two paragraphs constitute a review of those paragraphs of the Plane Geometry.

542. a. A Variable is a number which assumes different values during a particular discussion.

Thus a number x which assumes successively the values $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots$ is a *decreasing variable*, assuming ultimately values which differ by as little as we please from zero.

A number y which assumes successively the values $1, 1\frac{1}{2}, 1\frac{3}{4}, 1\frac{7}{8}, \dots$ is an *increasing variable*, assuming ultimately values which differ by as little as we please from 2.

b. A Constant is a number which has a fixed value throughout a particular discussion.

c. A Limit of a Variable is a constant such that the numerical value of the difference between the constant and the variable becomes and remains less than any small positive number. We say that a *variable approaches its limit*. If a variable has a limit, it has only one limit. The symbol $\doteq$ means "approaches the limit."

Thus, x above $\doteq 0$, and $y \doteq 2$.

543. Limits Theorems. It can be proved that:

a. If a variable x approaches a finite limit l , then cx , where c is a constant, approaches the limit cl . § 403, *a*

b. If two variables are constantly equal and each approaches a finite limit, then their limits are equal. § 403, *b*

c. If a variable x approaches a finite limit a , and a variable y approaches a limit b , then:

$(x \pm y)$ approaches the limit $a \pm b$;

xy approaches the limit $a \cdot b$;

$x \div y$ approaches the limit $a \div b$, provided b is not zero; and $\sqrt{xy}$ approaches the limit $\sqrt{ab}$.

Ex. 21. Find the length of the diagonal of a rectangular parallelepiped whose dimensions are 8, 9, and 12.

Ex. 22. Prove that the square of a diagonal of a rectangular parallelepiped is equal to the sum of the squares of its edges.

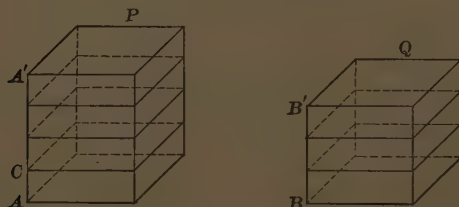
Ex. 23. Determine the length of the diagonal of a cube whose edge is of length s .

Note. — Supplementary Exercises 18–22, p. 456, can be studied now.

PROPOSITION V. THEOREM

544. *Two rectangular parallelopipeds having congruent bases have the same ratio as their altitudes.*

CASE I. *When the altitudes are commensurable.*



Hypothesis. *P and Q are rectangular parallelopipeds with congruent bases, and commensurable altitudes AA' and BB' .*

Conclusion.
$$\frac{Q}{P} = \frac{BB'}{AA'}.$$

Proof. 1. Let AC , a common measure of AA' and BB' , be contained 4 times in AA' and 3 times in BB' .

$$\therefore \frac{BB'}{AA'} = \frac{3}{4}.$$

2. Through the points of division of AA' and BB' draw planes $\perp AA'$ and BB' , respectively.

3. Then P is divided into 4, and Q is divided into 3, parts which are all congruent. § 536

4.
$$\therefore \frac{Q}{P} = \frac{3}{4}.$$

5.
$$\therefore \frac{Q}{P} = \frac{BB'}{AA'}.$$

Why?

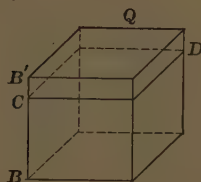
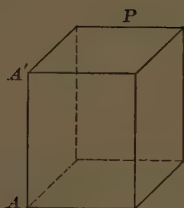
CASE II (Fig. p. 357). *When the altitudes are incommensurable.**

Hypothesis. *P and Q are rectangular parallelopipeds with congruent bases and incommensurable altitudes AA' and BB' .*

* This proof may be omitted if desired.

Conclusion.

$$\frac{Q}{P} = \frac{BB'}{AA'}$$



Proof. 1. Divide AA' into any number of equal parts, and apply one of these parts to BB' as unit of measure. Since AA' and BB' are incommensurable, a certain number of these parts will extend from B to C , leaving a remainder CB' less than the unit of measure.

2. Draw plane $CD \perp BB'$, and let parallelopiped BD be denoted by Q' .

Then

$$\frac{Q'}{P} = \frac{BC}{AA'}$$

Case I

[Since AA' and BC are commensurable.]

3. Let the number of subdivisions in AA' be indefinitely increased. The length of each subdivision will then diminish indefinitely, and hence CB' will approach the limit 0. § 542, c

Also $\frac{Q'}{P}$ will approach the limit $\frac{Q}{P}$, § 543, a

and $\frac{BC}{AA'}$ will approach the limit $\frac{BB'}{AA'}$. § 543, a

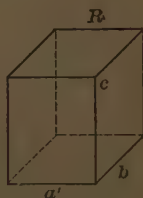
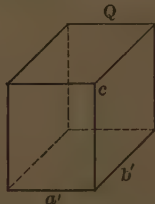
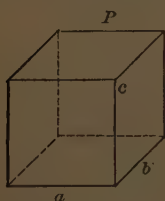
4. $\therefore \frac{Q}{P} = \frac{BB'}{AA'}$. § 543, b

545. Cor. If two rectangular parallelopeds have two dimensions of one equal respectively to two dimensions of the other, their volumes have the same ratio as their third dimensions.

Ex. 24. Compare the volumes of two rooms having the same floor space if their heights are 8' 6'' and 9' respectively.

PROPOSITION VI. THEOREM

546. *Two rectangular parallelopipeds having equal altitudes have the same ratio as their bases.*



Hypothesis. Rectangular parallelopipeds P and Q have equal altitudes c and bases with dimensions a , b , and a' , b' , respectively.

Conclusion.

$$\frac{P}{Q} = \frac{a \times b}{a' \times b'}$$

Proof. 1. Let R be a rectangular parallelopiped with dimensions c , a' , and b .

2. Then

$$\frac{P}{R} = \frac{a}{a'}$$

§ 545

and

$$\frac{R}{Q} = \frac{b}{b'}$$

Why?

3. Then, multiplying the equations of step 2,

$$\frac{P}{Q} = \frac{ab}{a'b'}$$

AX. 5, § 51

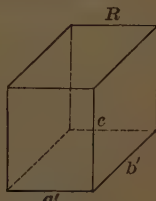
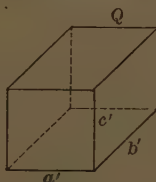
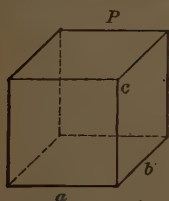
Note.—Two rectangular parallelopipeds having a dimension of one equal to one dimension of the other, have the same ratio as the products of their other two dimensions.

Ex. 25. Two rectangular parallelopipeds, with equal altitudes, have the dimensions of their bases 6 and 14, and 7 and 9, respectively. Find the ratio of their volumes.

Ex. 26. Compare each of the following rectangular parallelopipeds with each of the others: R , having dimensions 5, 7, and 9; S , having dimensions 9, 5, and 4; T , having dimensions 4, 6, and 7.

PROPOSITION VII. THEOREM

547. *Two rectangular parallelopipeds have the same ratio as the products of their three dimensions.*



Hypothesis. P and Q are rectangular parallelopipeds having dimensions a, b, c , and a', b', c' , respectively.

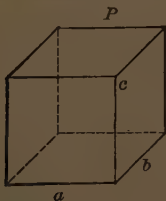
Conclusion.
$$\frac{P}{Q} = \frac{abc}{a'b'c'}$$

Suggestions.—1. Let R be a rectangular parallelopiped with dimensions a', b' , and c .

2. Find $\frac{P}{R}$ and $\frac{R}{Q}$ by § 546 and § 545. Then multiply $\frac{P}{R}$ by $\frac{R}{Q}$.

PROPOSITION VIII. THEOREM

548. *If the unit of solid measure is the cube whose edge is the linear unit, the volume of a rectangular parallelopiped equals the product of its three dimensions.*



Hypothesis. a, b , and c are the dimensions of rectangular parallelopiped P , and Q is the unit of measure.

Conclusion. Volume of $P = a \times b \times c$.

Proof. 1. Volume of $P = \frac{P}{Q}$. § 533

Complete the proof, applying § 547.

549. Cor. 1. *The volume of a cube is equal to the cube of its edge.*

550. Cor. 2. *The volume of a rectangular parallelopiped is equal to the product of its base and altitude.*

Note. — Corollaries 1 and 2 are expressed in their commonly abbreviated form. Expressed more accurately, the second would be “the *volume* of a rectangular parallelopiped is equal to the product of the *area* of the base and the *length* of the altitude.” The brief form of statement will be employed in the remaining theorems of solid geometry.

Remember that *volume* of a solid means the *number* of cubic units in it. In all succeeding theorems relating to volumes, it is understood that the *unit of solid* is the cube whose edge is the linear unit, and the *unit of surface* the square whose side is the linear unit.

Ex. 27. Find the ratio of the volumes of two rectangular parallelopipeds whose dimensions are 8, 12, and 21, and 14, 15, and 24, respectively.

Ex. 28. Find the volume and the area of the entire surface of a cube whose edge is 4 in.

Ex. 29. (a) If the edge of a cube is e , express by formulæ the total area, the volume, and the length of a diagonal. (b) Using the proper formula, determine the edge when the diagonal is 12. (c) Using the proper formula, determine the edge when the total area is 150 sq. in.

Ex. 30. Find the altitude of a rectangular parallelopiped, the dimensions of whose base are 21 and 30, equal to a rectangular parallelopiped whose dimensions are 27, 28, and 35.

Ex. 31. What must be the height of a tank having the form of a rectangular parallelopiped whose base has the dimensions 5 ft. and 8 ft., in order that the tank will contain 1800 gal. of water when the water rises to within one foot of the top? (One cu. ft. of water = $7\frac{1}{2}$ gal. approximately.)

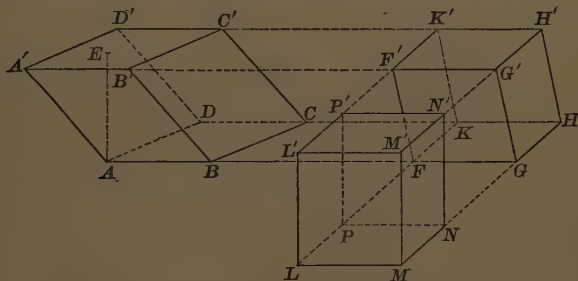
Ex. 32. How many barrels of water will run into a cistern during a $\frac{1}{2}$ in. fall of rain from the roof of a barn whose total roof area is 800 sq. ft. (One cu. ft. of water = $7\frac{1}{2}$ gal.)

Ex. 33. (a) How many cubic yards of concrete are required for the foundation walls of a house 25 ft. $\times$ 35 ft., if the walls are 10 in. thick and are 8 ft. high? (b) How many bags of cement are required if the mixture contains 4 bags of cement to one yard of gravel?

Note. — Supplementary Exercises 23–28, p. 456, can be studied now.

PROPOSITION IX. THEOREM

551. *The volume of any parallelopiped is equal to the product of its base and altitude.*



Hypothesis. AC' is an oblique parallelopiped.

Let the length of altitude AE be H , the area of base $ABCD$ be B , and the volume of AC' be V .

Conclusion. $V = BH$.

Proof. 1. Extend edges AB , $A'B'$, $D'C'$, and DC .

On AB extended, take $FG = AB$.

Draw planes FK' and $GH' \perp FG$, forming right parallelopiped FH' .

2. $\therefore$ prism $FH' =$ prism AC' . § 535

3. Extend edges HG , $H'G'$, $K'F'$, and KF .

On HG extended, take $NM = HG$; draw planes NP' and $ML' \perp NM$, forming rectangular parallelopiped LN' .

4. $\therefore$ prism $L'N =$ prism FH' . Why?

5. $\therefore$ prism $L'N =$ prism AC' . Why?

6. But volume $L'N = LMNP \times MM'$.

7. $\therefore$ volume $AC' = LMNP \times MM'$.

8. But the length of $MM' = H$

and area $LMNP =$ area $ABCD = B$. Note, p. 362

9. $\therefore$ vol. $AC' = BH$.

Note. — The student's understanding of this theorem will be increased greatly if a model of the above figure is at hand.

Note 1. — *Proof that LN' (step 3, § 551) is a rectangular parallelopiped.*

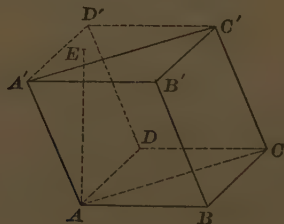
1. Since $FG \perp$ plane GH' , $\therefore$ plane $LH \perp$ plane MH' . § 495
2. Since $MM' \perp MN$, $\therefore MM' \perp$ plane LH . § 496
3. $\therefore \angle LMM' =$ a rt. $\angle$.
4. $\therefore LM'$ is a rectangle. § 141
5. $\therefore LN'$ is a rectangular parallelopiped. § 537

Note 2. — *Proof that $LMNP = ABCD$.*

$LMNP = FGHK$, since $\square$ s having equal bases and equal altitudes are equal. Similarly $FGHK = ABCD$, and $\therefore LMNP = ABCD$.

PROPOSITION X. THEOREM

552. *The volume of a triangular prism is equal to the product of its base and altitude.*



Hypothesis. $C'-ABC$ is a triangular prism.

Length of altitude $AE = H$; area of $\triangle ABC = B$; volume of $C'-ABC = V$.

Conclusion.

$$V = BH.$$

Suggestions. — 1. Consider the parallelopiped $D'-ABCD$, having its edges parallel to AB , BC , and BB' respectively.

2. Compare volume of $C'-ABC$ with that of $D'-ABCD$.

3. Express the volume of $D'-ABCD$, and then of $C'-ABC$.

Note. — At this point, the pupil should memorize the following formulæ if they are not already known.

1. Area of a $\triangle = \sqrt{s(s-a)(s-b)(s-c)}$, § 335
where the sides are a , b , and c , and $s = \frac{1}{2}(a+b+c)$.

2. Area of an equilateral $\triangle$ of side $s = \frac{s^2\sqrt{3}}{4}$. (Ex. 29, p. 199, Book IV.)

3. Altitude of an equilateral $\triangle$ of side $s = \frac{s\sqrt{3}}{2}$.

Ex. 34. Find the volume of a regular triangular prism the side of whose base is 5 and whose altitude is 10.

Ex. 35. Derive a formula for the volume of a regular triangular prism the side of whose base is s and whose altitude is h .

Ex. 36. Find the lateral area and volume of a right triangular prism, having the sides of its base 4, 7, and 9, respectively, and the altitude 8.

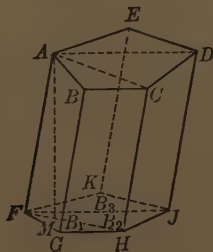
Suggestion. — To determine the area of the base, use the first formula in the note of § 552.

Ex. 37. Prove that the volume of a right triangular prism is equal to the product of the area of any face and one half the altitude to that face.

Ex. 38. The volume of a regular triangular prism is $96\sqrt{3}$, and one side of its base is 8. Find its lateral area.

PROPOSITION XI. THEOREM

553. *The volume of any prism is equal to the product of its base and altitude.*



Hypothesis. Let H = the length of altitude AM , B = the area of base $FGHJK$, and V = the volume of prism AJ .

Conclusion. $V = BH$.

Suggestions. — 1. Through edge AF and diagonals FH and FJ of the base, pass planes $AFHC$ and $AFJD$ dividing prism P into triangular prisms P_1 , P_2 , and P_3 , whose base areas are B_1 , B_2 , and B_3 , respectively, and whose common altitude is of length H .

2. Express the volumes of P_1 , P_2 , and P_3 . Add the results and simplify, thus obtaining an expression for the volume P .

554. Cor. 1. *Two prisms having equal bases and equal altitudes are equal.*

Suggestion.—Let prisms P and P' have equal bases B and B' respectively, and equal altitudes H and H' . Prove $P = P'$.

555. Cor. 2. *Two prisms having equal altitudes have the same ratio as their bases.*

556. Cor. 3. *Two prisms having equal bases have the same ratio as their altitudes.*

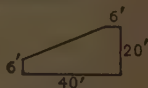
557. Cor. 4. *Two prisms have the same ratio as the products of their bases and altitudes.*

Ex. 39. Find the volume of a regular hexagonal prism the side of whose base is 4 in. and whose altitude is 9 in.

Ex. 40. Express by formulae the total area and the volume of a regular hexagonal prism whose base edge is e and whose height is h .

Ex. 41. A contractor agreed to dig a cellar at 35¢ per cubic yard. The lot was located upon a hillside so that the depth of the cellar at the back was 9 ft. and in front 5 ft. If the cellar was 38 ft. from the front to back, and was 25 ft. wide, how much did the contractor receive?

Ex. 42. How many cubic yards of concrete are required for a retaining wall 2 ft. thick whose dimensions are indicated on the adjoining figure?



Note.—Supplementary Exercises 29–33, p. 457, can be studied now.

PYRAMIDS

558. A **Pyramid** is a polyhedron bounded by three or more triangular faces which have a common vertex, and one other plane face, the **Base**, which intersects each of the triangular faces.

The common vertex of the triangular faces is the **Vertex** of the pyramid; the triangular faces are the **Lateral Faces**; the edges terminating at the vertex are the **Lateral Edges**; the sum of the areas of the lateral faces is the **Lateral Area**; the perpendicular from the vertex to the plane of the base is the **Altitude**.



559. A pyramid is called *triangular*, *quadrangular*, etc., according as its base is *triangular*, *quadrangular*, etc.

A **Regular Pyramid** is a pyramid whose base is inclosed by a regular polygon, and whose vertex lies in the perpendicular to the base at the center of the base.



A **Truncated Pyramid** is the part of a pyramid included between its base and a plane cutting all the lateral edges.

The base of the pyramid and the section of the cutting plane are called the *bases of the truncated pyramid*.

560. A **Frustum of a Pyramid** is a truncated pyramid whose bases are parallel.

The **Altitude** of a frustum is the perpendicular between the planes of the bases.



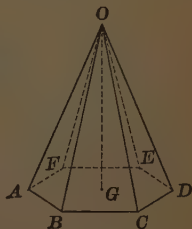
561. The following **important facts** about **pyramids** should be proved by the pupil:

I. *The lateral edges of a regular pyramid are equal.*

II. *The lateral faces of a regular pyramid are inclosed by congruent isosceles triangles.*

III. *The lateral faces of a frustum of any pyramid are inclosed by trapezoids.*

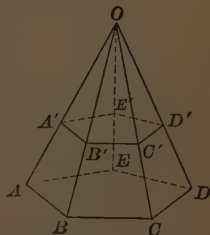
IV. *The lateral faces of a frustum of a regular pyramid are inclosed by congruent trapezoids.*



Suggestion. — Superpose $\triangle OAB$ on $\triangle OBC$.

Prove $ABB'A' \cong BCC'B'$.

V. *The lateral edges of a frustum of a regular pyramid are equal.*



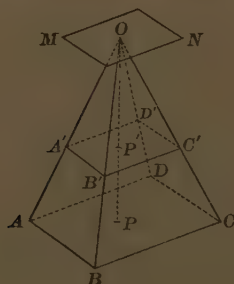
Note. — It will be assumed that the boundary of the base of the pyramid is a convex polygon.

PROPOSITION XII. THEOREM

562. *If a pyramid is cut by a plane parallel to the base :*

I. *The lateral edges and the altitude are divided proportionally.*

II. *The section is similar to the base.*



Hypothesis. Plane $A'B'C'D'$, parallel to the base of pyramid $O-ABCD$, intersects faces OAB , OBC , OCD , and ODA in lines $A'B'$, $B'C'$, $C'D'$, and $D'A'$, respectively, and altitude OP at P' .

Conclusion. I. $\frac{OA'}{OA} = \frac{OB'}{OB} = \frac{OD'}{OD} = \frac{OP'}{OP}$.

II. $A'B'C'D' \sim ABCD$.

Proof of I. 1. Through O , pass plane $MN \parallel ABCD$.

2. $\therefore \frac{OA'}{OA} = \frac{OB'}{OB} = \frac{OC'}{OC} = \frac{OD'}{OD} = \frac{OP'}{OP}$. § 482

Proof of II. 1. $\angle A'B'C' = \angle ABC$, $\angle B'C'D' = \angle BCD$, etc. Prove it. § 481

2. $\frac{A'B'}{AB} = \frac{OA'}{OA}$, $\frac{B'C'}{BC} = \frac{OB'}{OB}$, etc. Why?

3. $\therefore \frac{A'B'}{AB} = \frac{B'C'}{BC} = \frac{C'D'}{CD} = \frac{D'A'}{DA}$. Why?

4. $\therefore A'B'C'D' \sim ABCD$. Why?

563. Cor. 1. *The area of a section of a pyramid parallel to the base is to the area of the base as the square of its distance from the vertex is to the square of the altitude of the pyramid.*

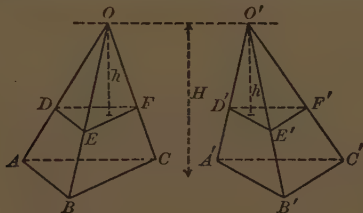
Proof. 1.
$$\frac{\text{Area } A'B'C'D'}{\text{Area } ABCD} = \frac{A'B'^2}{AB^2}.$$

§ 344

2. But
$$\frac{A'B'}{AB} = \frac{OA'}{OA} = \frac{OP'}{OP}.$$

3.
$$\therefore \frac{\text{Area } A'B'C'D'}{\text{Area } ABCD} = \frac{OP'^2}{OP^2}.$$

564. Cor. 2. *If two pyramids have equal altitudes and equal bases, sections parallel to the bases at equal distances from the vertices are equal.*



Hypothesis. Pyramids $O-ABC$ and $O'-A'B'C'$ have the common altitude H , and equal bases, ABC and $A'B'C'$.

DEF and $D'E'F'$ are sections of the pyramids parallel to the bases at the distance h from the vertices O and O' , respectively.

Conclusion. $DEF = D'E'F'.$

Proof. 1.
$$\frac{\text{Area } DEF}{\text{Area } ABC} = \frac{h^2}{H^2} \text{ and } \frac{\text{Area } D'E'F'}{\text{Area } A'B'C'} = \frac{h^2}{H^2}.$$

§ 563

Complete the proof.

Ex. 43. Prove that the areas of sections of a pyramid made by planes parallel to the base have the same ratio as the squares of the distances to the planes from the vertex.

Ex. 44. The altitude of a pyramid is 12 in., and its base is a square 9 in. on a side. What is the area of a section parallel to the base, whose distance from the vertex is 8 in.?

Ex. 45. What part of the area of the base of a pyramid is the area of a section made by a plane which is parallel to the base and bisects the altitude?

Note. — Supplementary Exercises 34–37, p. 457, can be studied now.

565. The **slant height** of a regular pyramid is the altitude of any lateral face. (See II, § 561.)

The **slant height** of a frustum of a regular pyramid is the altitude of any lateral face. (See III, § 561.)

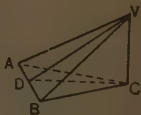
Ex. 46. Prove that the perimeter of the mid-section of a frustum of a pyramid is one half the sum of the perimeters of the bases.

Ex. 47. What is the slant height of a regular quadrangular pyramid whose altitude is 12 and the side of whose base is 4?

Suggestions.—1. Let ABV be one face, VC be the altitude, and C the center of the base.

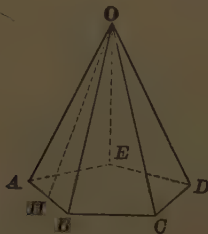
2. Determine the length of AC , and of DC ; then of VD .

Ex. 48. What is the slant height of a regular hexagonal pyramid whose altitude is 10 and whose base edge is 4?



PROPOSITION XIII. THEOREM

566. The lateral area of a regular pyramid is equal to the perimeter of its base multiplied by one half its slant height.



Hypothesis. $O-ABCDE$ is a regular pyramid.

P = perimeter of its base; L = length of its slant height;

S = the lateral area.

Conclusion.

$$S = \frac{1}{2} PL.$$

Proof. 1. Area of $\triangle OAB = \frac{1}{2} L \times AB$.

Why?

Complete the proof.

567. Cor. *The lateral area of the frustum of a regular pyramid is equal to one half the sum of the perimeters of the bases multiplied by the slant height.*



Hypothesis. AD' is a frustum of a regular pyramid.

L = the length of the slant height ; p and P = the perimeters of the upper and lower bases respectively ; and S = the lateral area.

Conclusion. $S = \frac{1}{2} L(p + P).$

Proof. 1. Area of $ABB'A' = \frac{1}{2} L(AB + A'B').$

Why ?

Complete the proof.

Ex. 49. What is the slant height of a regular triangular pyramid whose altitude is 10 and whose base edge is 4 ?

Suggestion.—Recall § 172.

Ex. 50. Express the lateral area of a regular pyramid in terms of the length of the slant height and the perimeter of the section midway between the base and the vertex.

Ex. 51. Prove the lateral surface of any pyramid greater than its base, when the perpendicular from the vertex to the base falls within the base.

Suggestion.—From the foot of the altitude draw lines to the vertices of the base ; each Δ formed has a smaller altitude than the corresponding lateral face.

Ex. 52. Determine the lateral area of each of the pyramids in Exercises 47 to 49.

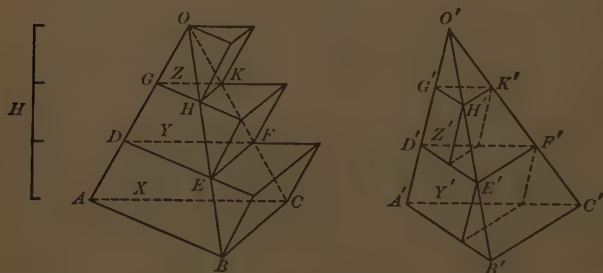
Ex. 53. In each of the Exercises 47 to 49 pass a plane parallel to the base at a distance of 5 in. from the vertex. Determine the lateral areas of the resulting frustums.

Ex. 54. Determine the total areas of the pyramids of Exercises 47 to 49.

Ex. 55. The edges of the bases of a frustum of a regular square pyramid are 5 in. and 10 in. respectively, and the altitude is 6 in. Determine the slant height and then the lateral area.

PROPOSITION XIV. THEOREM

568. *Two triangular pyramids having equal altitudes and equal bases are equal.*



Hypothesis. $O-ABC$ and $O'-A'B'C'$ have equal altitudes and equal bases ABC and $A'B'C'$.

Conclusion. $O-ABC = O'-A'B'C'$.

Proof. 1. Place the pyramids with their bases in the same plane, and let H represent their common altitude.

2. Divide H into 3 equal parts.

Through the points of division pass planes $\parallel$ to the plane of the bases, cutting $O-ABC$ in sections DEF and GHK , and $O'-A'B'C'$ in sections $D'E'F'$ and $G'H'K'$.

3. $\therefore DEF = D'E'F'$
and $GHK = G'H'K'$.

§ 564

4. With ABC , DEF , and GHK as *lower* bases, construct prisms X , Y , and Z with their lateral edges equal and $\parallel$ to AD ; with $D'E'F'$ and $G'H'K'$ as *upper* bases, construct prisms Y' and Z' , with their lateral edges equal and $\parallel$ to $A'D'$.

5. $\therefore$ prism $Y =$ prism Y'
and prism $Z =$ prism Z' .

Why?

5. Hence the sum of the prisms *circumscribed* about $O-ABC$ exceeds the sum of the prisms *inscribed* in $O-A'B'C'$ by prism X .

6. Evidently, $O-ABC < X + Y + Z$,
and $O-ABC > Y' + Z'$.

Likewise $O'-A'B'C'' < X + Y + Z$ and $> Y' + Z'$.

7. $\therefore O-ABC$ and $O'-A'B'C''$ differ by less than the difference between the sum of the circumscribed prisms and the sum of the inscribed prisms;

i.e. $O-ABC$ and $O'-A'B'C''$ differ by less than the lower prism X .

8. By increasing indefinitely the number of subdivisions of H , the volume of X can be made less than any assigned number, however small.

9. Suppose now that the volume of $O-ABC$ and $O'-A'B'C''$ differ by any amount k .

Since $X >$ the difference between $O-ABC$ and $O'-A'B'C''$, then X would be $> k$.

10. But this contradicts step 8.

11. $\therefore O'-A'B'C''$ and $O-ABC$ cannot differ at all;

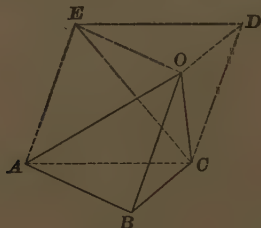
i.e. $O'-A'B'C'' = O-ABC$.

Note. — An interesting and instructive exercise at this point is that of proving the equality of two triangles which have equal bases and altitudes, by a proof like that given for Proposition XIV. In fact, it aids in understanding Proposition XIV if the exercise proposed is studied before taking up § 568.

The two triangles are compared with two sets of parallelograms, one inscribed in one triangle, the other circumscribed about the other triangle. The resulting figures are like the triangles AOB and $A'O'B'$ of the figure of § 568.

PROPOSITION XV. THEOREM

569. *The volume of a triangular pyramid is equal to one third the product of its base and altitude.*



Hypothesis. H = the length of the altitude; B = the area of the base; and V = the volume of pyramid $O-ABC$.

Conclusion. $V = \frac{1}{3} HB$.

Proof. 1. Let $EOD-ABC$ be the triangular prism having base ABC , and its lateral edges equal and parallel to OB .

2. Prism $EOD-ABC$ is composed of pyramid $O-ABC$ and pyramid $O-ACDE$.

3. Divide $O-ACDE$ into two triangular pyramids, $O-ACE$ and $O-CDE$ by passing a plane through E , O , and C .

4. In pyramids $O-ACE$ and $O-ECD$:

The altitudes are common.

Why?

Base ACE = base ECD .

Why?

$\therefore O-ACE = O-ECD$.

Why?

5. Pyramid $O-ECD$ is the same as pyramid $C-EOD$.

6. In pyramids $O-ABC$ and $C-EOD$:

The altitudes are equal.

Why?

Base ABC = base EOD .

Why?

$\therefore O-ABC = C-EOD$.

Why?

7. $\therefore O-ABC = O-ECD = O-ACE$.

8. $\therefore O-ABC = \frac{1}{3}$ prism $EOD-ABC$.

9. The altitude of prism $EOD-ABC = H$, and base = B .

10. $\therefore$ vol. $EOD-ABC = HB$.

Why?

11. $\therefore$ vol. $O-ABC = \frac{1}{3} HB$.

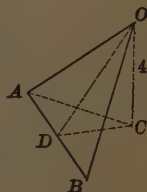
Ex. 56. If the base of a pyramid is a parallelogram, the plane determined by the vertex of the pyramid and a diagonal of the base divides the pyramid into two equal triangular pyramids.

Ex. 57. Determine the ratio to a given parallelopiped of the pyramid whose lateral edges are the three edges of the parallelopiped which intersect at any one corner.

Ex. 58. Each side of the base of a regular triangular pyramid is 6, and its altitude is 4. Find its lateral edge, lateral area, and volume.

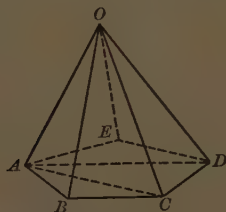
Suggestion. — In the figure, C is the center of the base.

Ex. 59. Find the area of the entire surface and the volume of a triangular pyramid, each of whose edges is 2.



PROPOSITION XVI. THEOREM

570. *The volume of any pyramid is equal to one third the product of its base and altitude.*



The proof is like that for § 553.

Proof to be given by the student.

571. Cor. 1. *Any two pyramids having equal bases and equal altitudes are equal.*

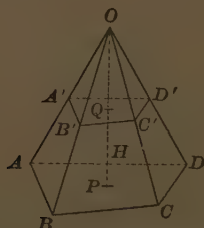
Cor. 2. *Two pyramids having equal altitudes have the same ratio as their bases.*

Cor. 3. *Two pyramids having equal bases have the same ratio as their altitudes.*

Cor. 4. *Any two pyramids have the same ratio as the products of their bases and altitudes.*

PROPOSITION XVII. THEOREM

572. *The volume of a frustum of any pyramid is equal to one third of its altitude multiplied by the sum of its upper base, its lower base, and the mean proportional between its bases.*



Hypothesis. V = the volume, B = the area of the lower base, b = the area of the upper base, and H = the length of the altitude of AC' , a frustum of any pyramid $O-AC$.

Conclusion. $V = \frac{1}{3} H (B + b + \sqrt{Bb})$.

Proof. 1. Draw altitude OP , cutting $A'C'$ at Q .

$$\begin{aligned} 2. \quad V &= \text{vol. } O-AC - \text{vol. } O-A'C' \\ &= \frac{1}{3} B (H + OQ) - \frac{1}{3} b (OQ) \\ &= \frac{1}{3} HB + \frac{1}{3} OQ (B - b). \end{aligned}$$

$$3. \text{ But } B : b = \overline{OP}^2 : \overline{OQ}^2. \quad \S 563$$

$$4. \quad \therefore \sqrt{B} : \sqrt{b} = OP : OQ. \quad \text{Algebra}$$

$$5. \quad \therefore (\sqrt{B} - \sqrt{b}) : \sqrt{b} = (OP - OQ) : OQ = H : OQ. \quad \S 256$$

$$6. \quad \therefore OQ (\sqrt{B} - \sqrt{b}) = H \sqrt{b}.$$

$$\begin{aligned} 7. \text{ Multiplying both members by } \sqrt{B} + \sqrt{b}, \\ \therefore OQ (B - b) = H (\sqrt{Bb} + b). \end{aligned}$$

8. Substituting in step 2 for $OQ (B - b)$ its value from step 7,

$$\begin{aligned} V &= \frac{1}{3} HB + \frac{1}{3} H (\sqrt{Bb} + b) \\ &= \frac{1}{3} H (B + b + \sqrt{Bb}). \end{aligned}$$

Ex. 60. Find the volume of a regular quadrangular pyramid each side of whose base is 3, and whose altitude is 5.

Ex. 61. Find the volume of a regular hexagonal pyramid each side of whose base is 4, and whose altitude is 9.

Ex. 62. The slant height and lateral edge of a regular quadrangular pyramid are 25 and $\sqrt{674}$, respectively. Find its lateral area and volume.

Ex. 63. Prove that the lines joining the center of a cube to the four vertices of one face are the edges of a regular quadrangular pyramid whose volume is $\frac{1}{6}$ that of the cube.

Ex. 64. Express the volume of a pyramid in terms of its altitude and the area of its mid-section parallel to the base.

Ex. 65. Find the lateral area and volume of a regular quadrangular pyramid, the area of whose base is 100, and whose lateral edge is 18.

Ex. 66. Find the area of the base of a regular quadrangular pyramid, whose lateral faces are equilateral triangles, and whose altitude is 5.

Suggestion. — Represent the lateral edge and the side of the base by x .

Ex. 67. Find the volume of a frustum of a regular quadrangular pyramid, the sides of whose bases are 9 and 5, respectively, and whose altitude is 10.

Ex. 68. Find the volume of a frustum of a regular triangular pyramid, the sides of whose bases are 18 and 6, respectively, and whose altitude is 24.

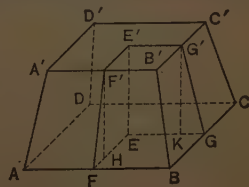
Ex. 69. Find the volume of a frustum of a regular hexagonal pyramid, the sides of whose bases are 8 and 4, respectively, and whose altitude is 12.

Ex. 70. A monument is in the form of a frustum of a regular quadrangular pyramid 8 ft. in height, the sides of whose bases are 3 ft. and 2 ft., respectively, surmounted by a regular quadrangular pyramid 2 ft. in height, each side of whose base is 2 ft. What is its weight, at 180 lb. to the cubic foot?

Ex. 71. The areas of the bases of a frustum of a pyramid are 12 and 75 respectively, and its altitude is 9. What is the altitude of the pyramid?

Suggestion. — Let the altitude of the pyramid = x ; then $x - 9$ is the $\perp$ from its vertex to the upper base of the frustum; then use § 563.

Ex. 72. The lateral edge of a frustum of a regular hexagonal pyramid is 10, and the sides of its bases are 10 and 4, respectively. Find its lateral area and volume.



Note. — Supplementary Exercises 38–42, p. 457, can be studied now.

SUPPLEMENTARY TOPICS

Five groups of supplementary material follow. None of this material is needed for subsequent parts of solid geometry. Each group is independent of each of the others. The groups are arranged in order of importance and of interest.

GROUP A. PRISMATOIDS

573. A **Prismatoid** is a polyedron bounded by two parallel faces called **Bases**, and by a number of lateral faces which are bounded by either triangles, trapezoids, or parallelograms, having one line in one of the bases of the prismatoid and the opposite vertex, or base, in the other base of the prismatoid.

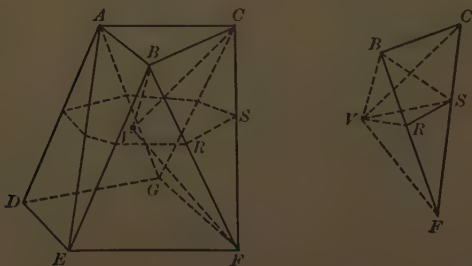
The **Altitude** of a prismatoid is the perpendicular between the bases.

The **Mid-section** of a prismatoid is the section of the plane parallel to the base and midway between them.

PROPOSITION XVIII. THEOREM

574. *If the areas of the lower and upper bases of a prismatoid are B and b , respectively, the area of the mid-section is m , the length of the altitude is H , and the volume is V , then,*

$$V = \frac{1}{6} H(B + b + 4m).$$



Proof. 1. Through any point V of the mid-section and each edge of the prismatoid, pass planes. These planes divide the

prismatoid into pyramid $V-ABC$, pyramid $V-DEFG$, pyramids like $V-BCF$, and polyedra like $V-ABED$.

(a) Volume $V-ABC = \frac{1}{3} \cdot \frac{1}{2} H \cdot b = \frac{1}{6} Hb$.

(b) Similarly, volume $V-DEFG = \frac{1}{6} HB$.

(c) To compute the volume of $V-BCF$:

1. Draw VR and VS , thus forming $\triangle VRS$, which is a part of the mid-section.

2. Draw plane VBS .

3. $V-BCF = V-RSF + V-BRS + V-BCS$.

4. $V-RSF = \frac{1}{3} \cdot \frac{1}{2} H \cdot \triangle VRS = \frac{1}{6} H \cdot \triangle VRS$. Prove it.

5. $V-BRS = \frac{1}{3} \cdot \frac{1}{2} H \cdot \triangle VRS = \frac{1}{6} H \cdot \triangle VRS$. Prove it.

6. $V-BCS = 2 \cdot V-BRS$, for $\triangle BCS = 2 \cdot \triangle BRS$. § 571, Cor. 2.

$$\therefore V-BCS = \frac{2}{6} H \cdot \triangle VRS.$$

7. $\therefore V-BCF = \frac{4}{6} H \cdot \triangle VRS$.

Similarly, the volume of any triangular pyramid with vertex V and as base a triangular lateral face is equal to $\frac{4}{6} H$ multiplied by that part of m which is in the triangular pyramid.

(d) $V-ABED$ can be divided into two triangular pyramids by passing a plane through V , A , and E . Hence its volume can be obtained as in part (c).

(e) Hence, the sum of all pyramids with vertex V , whose bases are lateral faces of the prismatoid, is $\frac{4}{6} H \cdot m$.

$$\begin{aligned} (f) \therefore \text{volume of the prismatoid} &= \frac{1}{6} HB + \frac{1}{6} Hb + \frac{4}{6} Hm \\ &= \frac{1}{6} H(B + b + 4m). \end{aligned}$$

Note. — This Proposition is particularly interesting not alone because it enables us to determine the volumes of many irregularly shaped figures, but because it includes many previous propositions as special cases.

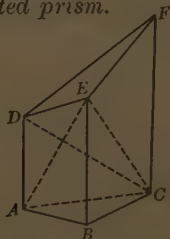
Ex. 73. Is a prism a special case of a prismatoid? In a prism, what relation is there between B , b , and m ? Does the formula of § 574 reduce to the usual formula for the volume of a prism?

Ex. 74. Answer the same questions for a pyramid.

GROUP B. TRUNCATED PRISMS

PROPOSITION XIX. THEOREM

575. *A truncated triangular prism is equal to the sum of three pyramids having as common base the lower base of the given prism, and having as their vertices the three vertices of the upper base of the truncated prism.*



Hypothesis. $DEF-ABC$ is a truncated triangular prism.

Conclusion. $DEF-ABC = E-ABC + D-ABC + F-ABC$.

Proof. 1. Pass planes through A, E , and C , and through D, E , and C , thus dividing $DEF-ABC$ into $E-ABC$, $E-ADC$, and $E-DFC$.

2. $E-ABC$ is one of the required pyramids.

3. $E-DAC = B-DAC$. Cor. 1, § 571

[For the altitudes from B and E to ADC are equal.]

But $B-DAC \equiv D-ABC$.

$\therefore E-DAC = D-ABC$, the second required pyramid.

4. $E-DFC = B-AFC$. § 571

[For $\triangle DFC = \triangle AFC$, and the altitudes from E and B to DFC and AFC , respectively, are equal.]

But $B-AFC \equiv F-ABC$.

$\therefore E-DFC = F-ABC$, the third required pyramid.

5. $\therefore DEF-ABC = E-ABC + D-ABC + F-ABC$.

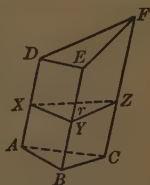
576. Cor. 1. *The volume of a truncated right triangular prism is equal to one third the base multiplied by the sum of the lateral edges.*

577. Cor. 2. *The volume of any truncated triangular prism is equal to the product of one third the area of a right section by the sum of the lateral edges.*

Suggestions. — 1. Let XYZ be the right section whose area is r .

2. Apply § 576, to DEF - XYZ , and to ABC - XYZ .

3. Prove DEF - $ABC = \frac{1}{3}r(AD + BE + CF)$.



Ex. 75. Find the volume of a truncated right triangular prism, the sides of whose base are 5, 12, and 13, and whose lateral edges are 3, 7, and 5, respectively.

Ex. 76. Find the volume of a truncated right triangular prism, whose lateral edges are 11, 14, and 17, having for its base an isosceles triangle whose sides are 10, 13, and 13, respectively.

Ex. 77. Find the volume of a truncated regular quadrangular prism, a side of whose base is 8, and whose lateral edges, taken in order, are 2, 6, 8, and 4, respectively.

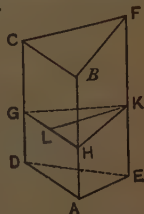
Suggestion. — Pass a plane through two diagonally opposite lateral edges, dividing the solid into two truncated right triangular prisms.

Ex. 78. If $ABCD$ is a rectangle, and EF is any line not in its plane parallel to AB , the volume of the solid bounded by figures $ABCD$, $ABFE$, $CDEF$, ADE , and BCF , is

$$\frac{1}{3}h \times AD \times (2AB + EF),$$

where h is the perpendicular from any point of EF to $ABCD$.

§ 577



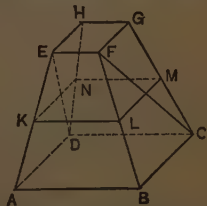
Ex. 79. If $ABCD$ and $EFGH$ are rectangles lying in parallel planes, AB and BC being parallel to EF and FG , respectively, the solid bounded by the figures $ABCD$, $EFGH$, $ABFE$, $BCGF$, $CDHG$, and $DAEH$, is called a *rectangular prismoid*.

$ABCD$ and $EFGH$ are called the bases of the rectangular prismoid, and the perpendicular distance between them, the altitude.

Prove the volume of a rectangular prismoid equal to the sum of its bases, plus four times a section equidistant from the bases, multiplied by one sixth the altitude.

Suggestion. — Pass a plane through CD and EF , and find the volumes of the solids $ABCD$ - EF and $EFGH$ - CD by Ex. 78.

Note. — Supplementary Exercises 43–47, p. 458, can be studied now.



GROUP C. MISCELLANEOUS THEOREMS

The following theorems about tetraedra are very much like certain theorems about triangles and about triedral angles.

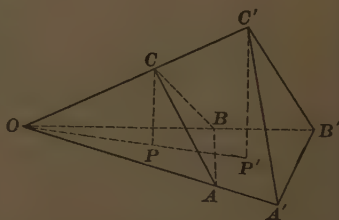
PROPOSITION XX. THEOREM

578. *Two tetraedra having a triedral angle of one equal to a triedral angle of the other, have the same ratio as the products of the edges including the equal triedral angles.*

Hypothesis. V and V' are volumes of tetraedra $O-ABC$ and $O-A'B'C'$, respectively, having the common triedral $\angle O$.

Conclusion.

$$\frac{V}{V'} = \frac{OA \times OB \times OC}{OA' \times OB' \times OC'}$$



Proof. 1. Draw lines CP and $C'P' \perp$ to face $OA'B'$.

2. Let their plane intersect face $OA'B'$ in line OPP' .

3. Now, OAB and $OA'B'$ are the bases, and CP and $C'P'$ the altitudes, of triangular pyramids $C-OAB$ and $C'-OA'B'$, respectively.

$$4. \quad \therefore \frac{V}{V'} = \frac{\text{area } OAB \times CP}{\text{area } OA'B' \times C'P'} \quad \text{Why?}$$

$$= \frac{\text{area } OAB}{\text{area } OA'B'} \times \frac{CP}{C'P'}. \quad (1)$$

$$5. \text{ But } \frac{\text{area } OAB}{\text{area } OA'B'} = \frac{OA \times OB}{OA' \times OB'}. \quad \S 346$$

6. Also $\triangle OCP$ and $OC'P'$ are rt. $\triangle$. Why?

7. Then $\triangle OCP$ and $OC'P'$ are similar. Why?

$$8. \quad \therefore \frac{CP}{C'P'} = \frac{OC}{OC'}. \quad \text{Why?}$$

9. Substituting from steps 5 and 8 in step 4,

$$\frac{V}{V'} = \frac{OA \times OB}{OA' \times OB'} \times \frac{OC}{OC'} = \frac{OA \times OB \times OC}{OA' \times OB' \times OC'}.$$

Ex. 80. State the theorem of plane geometry about triangles which corresponds to Proposition XX.

Note. — For each of the following exercises, state also the corresponding theorem about triangles.

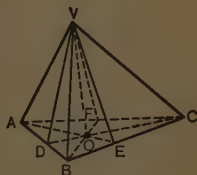
Ex. 81. Prove that two tetraedra are congruent if a dihedral angle and the adjacent faces of one are congruent, respectively, to a dihedral angle and the adjacent faces of the other, if the congruent parts are arranged in the same order.

Suggestion. — Prove by superposition.

Ex. 82. Two tetraedra are congruent if three faces of one are congruent, respectively, to three faces of the other, if the congruent parts are arranged in the same order.

Suggestion. — Recall § 515.

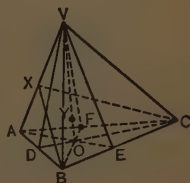
Ex. 83. Prove that the three planes passing through the lateral edges of a triangular pyramid, bisecting the sides of the base, meet in a common straight line.



Ex. 84. Prove that the six planes through the edges of a tetraedron bisecting the opposite edges meet in a common point.

Suggestion. — By Ex. 83, three planes meet in line VO. Let plane XBC intersect VO at Y. Prove Y lies in the remaining two planes.

Note. — The common point is the *center of gravity* of the tetraedron.



Ex. 85. Prove that the center of gravity of a tetraedron divides the line drawn from any vertex to the center of gravity of the opposite face in the ratio 3 : 1.

Ex. 86. Prove that the six planes bisecting the dihedral angles of a tetraedron meet in a common point.

Note. — Pupils will find it interesting to attempt to make up other theorems about tetraedra which are suggested by theorems about triangles.

GROUP D. REGULAR POLYEDRA

579. A **Regular Polyedron** is a polyedron whose faces are congruent regular polygons, and whose polyedral angles are all congruent.

PROPOSITION XXI. THEOREM

580. *Not more than five regular convex polyedra are possible.*

A convex polyedral $\angle$ must have at least three faces, and the sum of its face $\angle$ s must be $< 360^\circ$. § 514

1. *With equilateral triangles.*

Since each $\angle$ of an equilateral Δ is 60° , we may form a convex polyedral $\angle$ by combining either 3, 4, or 5 equilateral Δ .

Not more than 5 equilateral Δ can be combined to form a convex polyedral $\angle$. § 514

Hence not more than three regular convex polyedra can be bounded by equilateral Δ .

2. *With squares.*

Since each $\angle$ of a square is 90° , we may form a convex polyedral $\angle$ by combining 3 squares.

Not more than 3 squares can be combined to form a convex polyedral $\angle$.

Hence not more than one regular convex polyedron can be bounded by squares.

3. *With regular pentagons.*

Since each $\angle$ of a regular pentagon is 108° , we may form a convex polyedral $\angle$ by combining 3 regular pentagons.

Not more than 3 regular pentagons can be combined to form a convex polyedral $\angle$. Why?

Hence not more than one regular convex polyedron can be bounded by regular pentagons.

4. *With other regular polygons.*

Since each $\angle$ of a regular hexagon is 120° , no convex polyedral $\angle$ can be formed by combining regular hexagons. Why?

Hence no regular convex polyedron can be bounded by regular hexagons.

In like manner, no regular convex polyedron can be bounded by regular polygons of more than six sides.

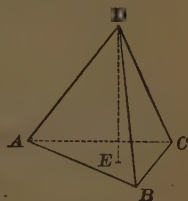
Therefore, not more than five regular convex polyedra are possible.

Ex. 87. Prove that the following construction produces a regular tetraedron having a given side.

Construction.—1. With given side AB , construct equilateral ABC .

2. At its center E , draw $ED \perp ABC$.

3. Take D on DE so that $AD = AB$, and draw AD , BD , and CD .



Statement.— $ABCD$ is a regular tetraedron.

Prove its faces are inclosed by congruent equilateral Δ s and that its triedral angles are congruent.

581. Models of the five regular polyedra may be made by drawing upon cardboard figures like the following.

Cut out the figures along the outer line; cut only halfway through the cardboard on the inner lines; bring the edges together and fasten them with gummed paper.



TETRAEDRON



HEXAEDRON



OCTAEDRON



DODECAEDRON



ICOSAEDRON

Ex. 88. Make a model of at least one of the regular polyedra.

Ex. 89. What is the sum of the face angles at any vertex of each of the regular polyedra?

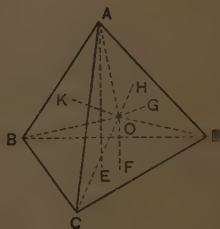
Ex. 90. Find the volume and the total area of a regular tetraedron whose edge is 10.

Ex. 91. Find the volume and the total area of a regular tetraedron whose edge is a .

Ex. 92. The sum of the perpendiculars drawn to the faces from any point within a regular tetraedron is equal to its altitude.

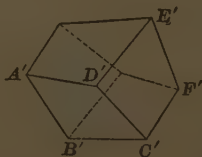
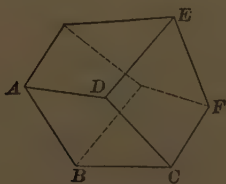
Suggestion.—Divide the tetraedron into triangular pyramids, having the given point for their common vertex. Find the volume of each and of the whole pyramid and form an equation.

Ex. 93. Prove that the volume of a regular octaedron is equal to the cube of its edge multiplied by $\frac{1}{6}\sqrt{2}$.



GROUP E. SIMILAR POLYEDRA

582. Two polyedra are similar when they have the same number of faces similar each to each and similarly placed, and have their homologous polyedral angles congruent.



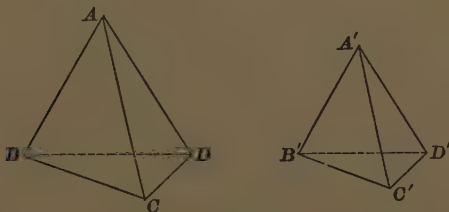
Ex. 94. Prove that the ratio of any two homologous edges of two similar polyedra is equal to the ratio of any other two homologous edges.

Ex. 95. Prove that any two homologous faces of two similar polyedra are to each other as the squares of any two homologous edges.

Ex. 96. Prove that the entire surfaces of two similar polyedra are to each other as the squares of any two homologous edges.

PROPOSITION XXII. THEOREM

583. *Two tetraedra are similar when three face triangles including a triedral angle of one are similar, respectively, to three face triangles including a triedral angle of the other, and similarly placed.*



Hypothesis. In tetraedra $ABCD$ and $A'B'C'D'$
 $\triangle ABC \sim \triangle A'B'C'$, $\triangle ACD \sim \triangle A'C'D'$, and
 $\triangle ADB \sim \triangle A'D'B'$.

Conclusion. $ABCD \sim A'B'C'D$.

Proof. 1. From the given similar $\triangle$ s, we have

$$\frac{BC}{B'C'} = \left(\frac{AC}{A'C'} \right) = \frac{CD}{C'D'} = \left(\frac{AD}{A'D'} \right) = \frac{BD}{B'D'}. \quad \text{Why?}$$

2. Hence, $\triangle BCD \sim \triangle B'C'D'$. Why?

3. Again, $\angle BAC$, CAD , and DAB are equal, respectively, to $\angle B'A'C'$, $C'A'D'$, and $D'A'B'$. Why?

4. Then, triedral $\angle A-BCD$ and $A'-B'C'D'$ are congruent.

§ 516

5. Similarly, any two homologous triedral $\angle$ s are congruent.

6. Therefore, $ABCD$ and $A'B'C'D'$ are similar.

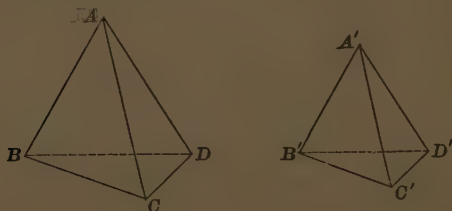
§ 582

Ex. 97. Two tetraedra are similar when a diedral angle of one is congruent to a diedral angle of the other, and the face triangles including the congruent diedral angles are similar each to each, and similarly placed.

Ex. 98. If a tetraedron be cut by a plane parallel to one of its faces, the tetraedron cut off is similar to the given tetraedron.

PROPOSITION XXIII. THEOREM

584. *Two similar tetraedra have the same ratio as the cubes of any two homologous edges.*



Hypothesis. V and V' are the volumes of similar tetraedra $ABCD$ and $A'B'C'D'$, respectively, A and A' being homologous vertices.

Conclusion.
$$\frac{V}{V'} = \frac{\overline{AB}^3}{\overline{A'B'}^3}.$$

Proof. 1. Since $ABCD \sim A'B'C'D'$
 triedral $\angle A = \text{triedral } \angle A'$.

2.
$$\therefore \frac{V}{V'} = \frac{AB \times AC \times AD}{A'B' \times A'C' \times A'D'}. \quad \S 578$$

3.
$$\therefore \frac{V}{V'} = \frac{AB}{A'B'} \times \frac{AC}{A'C'} \times \frac{AD}{A'D'}.$$

4. But $\frac{AC}{A'C'} = \frac{AB}{A'B'}$

and $\frac{AD}{A'D'} = \frac{AB}{A'B'}. \quad \text{Why?}$

5. Substituting from step 4 in step 3,

$$\frac{V}{V'} = \frac{AB}{A'B'} \times \frac{AB}{A'B'} \times \frac{AB}{A'B'}.$$

6.
$$\therefore \frac{V}{V'} = \frac{\overline{AB}^3}{\overline{A'B'}^3}.$$

Note 1. It can be proved that any two similar polyedra can be divided into the same number of tetraedra, similar each to each and similarly placed. Consequently, it can be proved that any two similar polyedra have the same ratio as the cubes of any two homologous edges.

Note 2. It is interesting to notice that in similar figures

(a) two homologous lines have the same ratio as any two homologous sides ;

(b) the areas of two homologous limited or bounded surfaces have the same ratio as the squares of any two homologous sides ;

(c) the volumes of two homologous solid parts have the same ratio as the cubes of any two homologous sides.

Ex. 99. The volume of a pyramid whose altitude is 7 in. is 686 cu. in. Find the volume of a similar pyramid whose altitude is 12 in.

Ex. 100. If the volume of a prism whose altitude is 9 ft. is 171 cu. ft., find the altitude of a similar prism whose volume is $50\frac{2}{3}$ cu. ft.

Ex. 101. Two bins of similar form contain, respectively, 375 and 648 bushels of wheat. If the first bin is 3 ft. 9 in. long, what is the length of the second ?

Ex. 102. A pyramid whose altitude is 10 in. weighs 24 lb. At what distance from its vertex must it be cut by a plane parallel to its base so that the frustum cut off may weigh 12 lb. ?

Ex. 103. An edge of a polyedron is 56, and the homologous edge of a similar polyedron is 21. The area of the entire surface of the second polyedron is 135, and its volume is 162. Find the area of the entire surface and the volume of the first polyedron.

Ex. 104. The area of the entire surface of a tetraedron is 147, and its volume is 686. If the area of the entire surface of a similar tetraedron is 48, what is its volume ?

Suggestion. — Let x and y denote the homologous edges of the tetraedra.

Ex. 105. The area of the entire surface of a tetraedron is 75, and its volume is 500. If the volume of a similar tetraedron is 32, what is the area of its entire surface ?

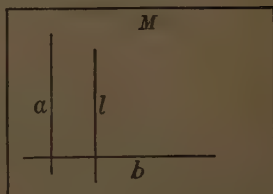
Ex. 106. The homologous edges of three similar tetraedra are 3, 4, and 5, respectively. Find the homologous edge of a similar tetraedron equivalent to their sum.

Suggestion. — Represent the edge by x .

BOOK VIII

THE CYLINDER AND THE CONE

585. Generating a Surface. If a straight line l moves so that it constantly intersects a straight line b and is constantly parallel to a straight line a which intersects b , it can be proved that l constantly lies in the plane M determined by a and b ; also it can be proved that every point in M lies in line l at some time during its period of movement.

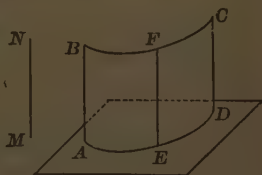


Line l is said to *generate the plane M* .

By suitable movement of a straight line, the straight line can be made to generate various curved surfaces as well as a plane surface.

586. A Cylindrical Surface is the surface generated by a moving straight line which constantly intersects a given plane curve and which is constantly parallel to another given straight line not in the plane of the curve.

Thus, if AB moves so as constantly to intersect plane curve AD and to be constantly parallel to MN , not in the plane of AD , then AB generates the cylindrical surface BD .



The moving line is called the **Generatrix**; the curved line is called the **Directrix**. Any position of the generatrix, as EF , is called an **Element** of the cylindrical surface.

Ex. 1. How many elements does a cylindrical surface have?

Ex. 2. Consider any two elements of a cylindrical surface. What kind of lines are they?

Ex. 3. Can more than one cylindrical surface be generated by use of a given directrix?

587. If the directrix is a *closed* plane curve, the cylindrical surface separates an infinite part of space from surrounding space.

The cylindrical surfaces considered in this text always have closed convex directrices.



588. A **Cylinder** is the solid (§ 525) bounded by a portion of a cylindrical surface and by portions of two parallel planes which intersect all the elements of the surface.

The portions of the parallel planes are the **Bases** of the cylinder; and the portion of the cylindrical surface between the planes is the **Lateral Surface** of the cylinder.



The **Total Surface** of a cylinder consists of its lateral surface and its bases.

The perpendicular between the two parallel planes is the **Altitude** of the cylinder. The segment of an element of the cylindrical surface which lies between the bases is an **Element** of the cylinder.

A **Section** of a cylinder is the part of the cylinder common to it and a plane cutting all the elements of the cylinder.

If a section of a cylinder is made by a plane perpendicular to the elements, it is a **Right Section** of the cylinder.

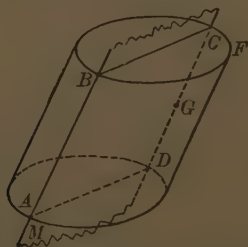
Ex. 4. The elements of a cylinder are equal and parallel.

589. Kinds of Cylinders. A **Right Cylinder** is a cylinder whose elements are perpendicular to its bases.

A **Circular Cylinder** is a cylinder whose base is inclosed by a circle.

PROPOSITION I. THEOREM

590. *If a plane passes through an element of a cylinder and through at least one other point of the surface of the cylinder, the intersection with the total surface of the cylinder is a parallelogram.*



Hypothesis. Plane M , passing through element AB of cylinder AF , intersects the lateral surface of AF in point G , not in AB .

Conclusion. The section of AF made by plane M is inclosed by a parallelogram.

Proof. 1. Through G draw $CD \parallel AB$.

2. $\therefore CD$ is an element of surface AF , intersecting the bases at C and D respectively. § 586

3. CD is also in M . § 447, I and III

4. $\therefore CD$ is in the intersection of AF and M .

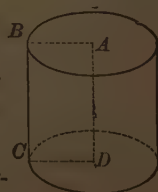
5. BC and AD are $\parallel$ straight lines, lying in both M and the bases of AF . Why?

6. $\therefore ABCD$ is the intersection of AF and M .

7. Also $ABCD$ is a $\square$. Why?

Ex. 5. If a rectangle revolves about one of its sides as an axis, it generates a right circular cylinder.

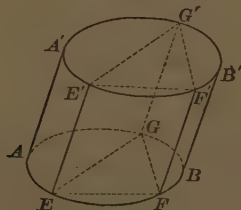
Suggestion. — Prove that C describes a circle, that BC generates a cylinder, and that the cylinder is a right cylinder.



591. As a consequence of Ex. 5, a right circular cylinder is also called a *cylinder of revolution*.

PROPOSITION II. THEOREM

592. *The bases of a cylinder are congruent.*



Hypothesis. AB' is any cylinder, with bases $A'B'$ and AB .

Conclusion. Base $A'B' \cong$ base AB .

Proof. 1. Let E' and F' be two particular points of curve $A'B'$ and G' any other point. Draw elements EE' , FF' , and GG' ; also draw EF , FG , EG , $E'F'$, $F'G'$, and $E'G'$.

2. $\therefore \triangle EFG \cong \triangle E'F'G'$. Prove it.

3. $\therefore$ base $A'B'$ may be superposed on base AB so that E' , F' , and G' will fall upon E , F , and G respectively.

4. But G' was any point of $A'B'$ except E' and F' .

5. $\therefore E'$, F' , and every other point of curve $A'B'$ will fall upon a corresponding point of curve AB .

6. $\therefore$ base $A'B' \cong$ base AB . Why?

Note.—It is obvious that every point of curve AB can be proved to fall upon a corresponding point of $A'B'$.

Ex. 6. Prove that a section of a circular cylinder made by a plane parallel to the base is inclosed by a circle.

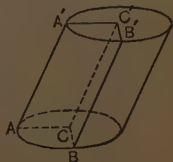
Note.—A section of a circular cylinder made by a plane not parallel to the base is inclosed by a curve called an *ellipse*.

Ex. 7. Prove that a line drawn parallel to the elements of a circular cylinder from the center of one base intersects the other base at its center.

Suggestions.—1. Let A' be a particular point and B' be any other point of the bounding circle of the upper base whose center is C' .

2. Draw $C'C \parallel B'B$, and draw element $A'A$.

3. Prove that $CB = CA$, and that C is the center of curve AB .



593. The **Axis** of a circular cylinder is a straight line drawn between the centers of its bases.

Ex. 8. Prove that the axis of a circular cylinder is parallel to the elements of the lateral surface.

Ex. 9. Prove that the axis of a circular cylinder passes through the centers of all sections parallel to the bases.

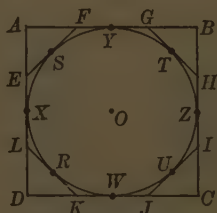
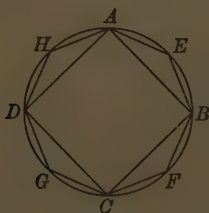
MEASURING THE CYLINDER

594. In measuring the cylinder, difficulties are encountered which are like those met in measuring a circle. For example, the ratio of the cylindrical surface to the customary unit of surface measure cannot have meaning in the ordinary sense since they are not magnitudes of the same kind, one being a plane and one a curved surface.

595. Application of Limits to the Circle.

(a) The circumference of a circle, *i.e.* the length of a circle, is defined to be the limit of the perimeter of any regular inscribed polygon as the number of sides is indefinitely increased.

§ 412



(b) It is proved that the perimeter of any regular circumscribed polygon (as well as inscribed polygon) approaches the circumference of the circle as limit if the number of sides is indefinitely increased.

§ 413

(c) It can be proved that the perimeter of *any* inscribed or circumscribed polygon approaches the circumference of the circle as limit if the number of sides is increased indefinitely in such manner that the length of every side approaches the limit zero.

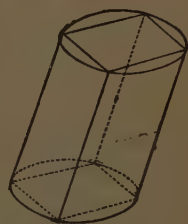
(d) The area of a circle is defined to be the limit of the area of any regular inscribed polygon as the number of sides increases indefinitely. § 417

(e) It is proved that the area of any regular circumscribed polygon approaches the area of the circle as limit as the number of sides is increased indefinitely. § 418

(f) It can be proved that the area of *any* inscribed or circumscribed polygon approaches the area of the circle as limit if the number of sides is increased indefinitely in such manner that the length of each side approaches the limit zero.

Note. — Remember that $\doteq$ is the symbol for “approaches the limit.”

596. A *prism* is *inscribed in a cylinder* when its lateral edges are elements of the cylinder and its bases are in the planes of the bases of the cylinder. The polygons bounding the bases of the prism are inscribed in the boundaries of the bases of the cylinder.



597. Application of Limits to a Cylinder.

Inscribe in a circular cylinder a prism having its base inclosed by a regular polygon; then inscribe a second prism whose base is inclosed by a regular polygon having double the number of sides; imagine that this process is continued indefinitely. Pass a plane forming right sections of the cylinder and the prisms.

It will be assumed evident that the prisms come nearer and nearer to occupying the same space as the cylinder. As a consequence:

(a) The **Volume of the Cylinder** is defined to be the limit of the volume of the inscribed prism as the number of faces increases indefinitely.

(b) The **Lateral Area of a Cylinder** is defined to be the limit of the lateral area of the inscribed prism as the number of faces increases indefinitely.

(c) The length of a right section of the lateral surface of the cylinder is defined to be the limit of the length of the right

section of the lateral surface of the inscribed prism as the number of faces increases indefinitely.

(d) It is evident that the edge and altitude of the inscribed prism equal respectively the element and altitude of the cylinder.

PROPOSITION III. THEOREM

598. *The lateral area of a circular cylinder is equal to the perimeter of a right section multiplied by the length of an element.*



Hypothesis. S = the lateral area, P = the perimeter of a right section, E = the length of an element of a circular cylinder.

Conclusion. $S = E \times P.$

Proof. 1. Inscribe in the cylinder a prism whose base is inclosed by a regular polygon.

Let S' = the lateral area and P' = the perimeter of a right section.

2. Then $S' = E \times P'.$ Why?

3. Let the number of faces of the prism increase indefinitely. Then

$S' \doteq S$, and $P' \doteq P.$ § 597, b and c ; also Note, § 595.

4. $\therefore E \times P' \doteq E \times P.$ § 543, a

5. $\therefore S = E \times P.$ § 543, b

599. Cor. 1. *The lateral area of a right circular cylinder is equal to the circumference of its base multiplied by the length of the altitude.*

600. Cor. 2. If R = the length of the radius of the base, H = the length of the altitude, S = the lateral area, and T = the total area of a right circular cylinder, then

$$(a) S = 2\pi RH.$$

$$(b) T = 2\pi R^2 + 2\pi RH = 2\pi R(R + H).$$

Ex. 10. Find the lateral area of a right circular cylinder whose altitude is 16 and the diameter of whose base is 18.

Ex. 11. Find the total area of a cylinder of revolution whose altitude is 15 and the radius of whose base is 5.

Ex. 12. Determine the cost at 15¢ per square yard of painting the vertical surface and top of a gas holder whose diameter is 30 ft. and whose height is 20 ft.

Ex. 13. How many square feet of tin are required to make 30 sections of hot-air furnace pipe 10 in. in diameter and 30 in. in length?

PROPOSITION IV. THEOREM

601. *The volume of a circular cylinder is equal to the area of its base multiplied by the length of its altitude.*

Hypothesis. V = the volume, B = the area of the base, and H = the length of the altitude of a circular cylinder.

Conclusion. $V = H \times B.$

Proof. 1. Inscribe in the cylinder a prism having its base inclosed by a regular polygon. Let V' = the volume and B' = the area of the base.

2. $\therefore V' = H \times B'.$ Why?

3. Increase indefinitely the number of faces of the prism. Then $V' \doteq V$, and $B' \doteq B.$

4. $\therefore H \times B' \doteq H \times B.$ § 543, a.

5. $\therefore V = H \times B.$ § 543, b.

602. Cor. If V = the volume, H = the length of the altitude, and R = the radius of the base of a circular cylinder, then

$$V = \pi R^2 H.$$

Ex. 14. What is the cost of digging a dry well 5 ft. in diameter and 15 ft. deep at 50 ¢ per cubic yard?

Ex. 15. What is the capacity in gallons of a water tank 12 ft. in length and 36 in. in diameter, estimating $7\frac{1}{2}$ gal. of water to a cubic foot?

Ex. 16. How many cubic feet of metal are there in a hollow cylindrical tube 18 ft. long, whose outer diameter is 8 in. and whose thickness is 1 in.?

Ex. 17. Determine the number of cubic yards of concrete required for the wall and floor of a circular cistern 8 ft. in outside diameter, and 12 ft. deep, if the walls and floor are 8 in. thick.

Ex. 18. Determine the diameter of a 2-bbl. water reservoir having the form of a right circular cylinder if the length is 4 ft. (2 bbl. = 63 gal.; 1 cu. ft. contains $7\frac{1}{2}$ gal.)

Note. — Supplementary Exercises 48–58, p. 459, can be studied now.

THE CONE

603. A **Conical Surface** is the surface generated by a moving straight line, which constantly intersects a given plane curve and constantly passes through a given point not in the plane of the curve.

Thus, if line OA moves so as constantly to intersect plane curve ABC , and constantly passes through point O , not in the plane of the curve, it generates a conical surface.

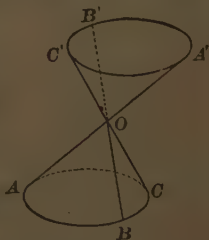
The moving line is called the **Generatrix**, and the curve the **Directrix**.

The given point is called the **Vertex**, and any position of the generatrix, as OB , is called an **Element** of the surface.

If the generatrix be supposed to be indefinite in length, it will generate two conical surfaces of indefinite extent, $O-A'B'C'$ and $O-ABC$.

These are called the *upper* and *lower nappes*, respectively, of the conical surface.

It will be assumed that the directrix is a closed plane curve so that each nappe of the surface separates an infinite portion of space from surrounding space.



604. A **Cone** is a solid bounded by a portion of one nappe of a conical surface and that part of a plane cutting all the elements of the surface which lies within the surface.

The plane is called the **Base** of the cone, and the conical surface the **Lateral Surface**.

The **Altitude** of a cone is the perpendicular from the vertex to the plane of the base.



605. Kinds of Cones.

A **Circular Cone** is a cone whose base is inclosed by a circle.

The **Axis** of a circular cone is a straight line drawn from the vertex to the center of the base.

A **Right Circular Cone** is a circular cone whose axis is perpendicular to its base.

A **Frustum of a Cone** is a portion of a cone included between the base and a plane parallel to the base.

The base of the cone is called the *lower base*, and the section made by the plane the *upper base, of the frustum*.

The *altitude of a frustum* is the perpendicular between the planes of the bases.

Ex. 19. Prove that the elements of a right circular cone are equal.

Ex. 20. Prove that the elements of a frustum of a right circular cone are equal.

Ex. 21. If a right triangle be revolved about one of its legs as an axis, it generates a right circular cone.

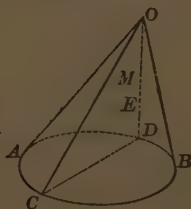
606. As a consequence of Ex. 19, the distance from the vertex to any point of the circle bounding the base of a right circular cone is called the **Slant Height** of the right circular cone.

As a consequence of Ex. 20, the portion of the slant height of a right circular cone between the base and a plane parallel to the base is called the **Slant Height** of the frustum of the right circular cone.

As a consequence of Ex. 21, a right circular cone is also called a cone of revolution.

PROPOSITION V. THEOREM

607. *If a plane passes through an element of a cone and through at least one other point of the surface of the cone, the intersection with the total surface of the cone is a triangle.*



Hypothesis. Plane M , passing through element OC of cone $O-AB$, intersects the surface again in point E , not in OC .

Conclusion. The intersection of M and the total surface of $O-AB$ is a triangle.

Proof. 1. Draw element OED , cutting the base AB in point D . § 604

2. $\therefore$ st. line CD lies in planes M and ACB . Why?

3. $\therefore \triangle OCD$ lies in M and in the total surface of $O-AB$, and is their intersection.

Note. — This theorem assumes that the curve ACB is such that a plane cuts it in only two points.

Ex. 22. Find, correct to one decimal, the slant height of a right circular cone whose altitude is 9 in. and the radius of whose base is 3 in.

Ex. 23. Find the slant height of a right circular cone whose altitude is h and the radius of whose base is r .

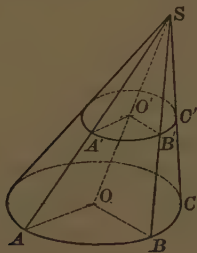
Ex. 24. The radii of the upper and lower bases of the frustum of a right circular cone are 3 in. and 5 in. respectively; the altitude of the frustum is 6 in. Determine the slant height correct to one decimal place.

Ex. 25. Find the altitude of a right circular cone whose slant height is 13 and the radius of whose base is 5.

Ex. 26. Repeat Ex. 24 when the radii are r and R respectively and the altitude is H .

PROPOSITION VI. THEOREM

608. *The section of the lateral surface of a circular cone made by a plane parallel to the base is a circle.*



Hypothesis. $A'B'C'$ is the section of circular cone $S-ABC$, made by a plane $\parallel$ base ABC whose center is O .

Conclusion. $A'B'C'$ is a $\odot$.

Proof. 1. Draw axis OS , intersecting $A'B'C'$ at O' .

2. Let A' and B' be any two points in curve $A'B'C'$. Let planes $A'O'S$ and $B'O'S$ intersect the base in radii OA and OB , the cutting plane in lines $O'A'$ and $O'B'$, and the lateral surface in lines $SA'A$ and $SB'B$ respectively.

3. $SA'A$ and $SB'B$ are straight lines. § 604

4. $\triangle SA'O' \sim \triangle SAO$, and $\triangle SB'O' \sim \triangle SBO$. Prove it.

5. $\therefore \frac{O'A'}{OA} = \frac{O'B'}{OB}$. Prove it.

6. $\therefore O'A' = O'B'$. Prove it.

7. Since A' and B' are any two points of curve $A'B'C'$ and are equidistant from O' , curve $A'B'C'$ is a circle and O' is its center.

609. Cor. *The axis of a circular cone passes through the center of every section parallel to the base.*

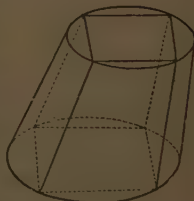
Ex. 27. Prove that the radii of the upper and lower bases of a frustum of any cone have the same ratio as the distances of the bases from the vertex of the cone.

MEASURING THE CONE

610. A **pyramid** is **inscribed in a cone** when its lateral edges are elements of the cone and the bounding polygon of the base of the pyramid is inscribed in the boundary of the base of the cone. The vertex and altitude of the pyramid coincide with the vertex and the altitude of the cone.



611. A **frustum of a pyramid** is **inscribed in a frustum of a cone** when its lateral edges are elements of the frustum of the cone, and the boundaries of the bases of the frustum of the pyramid are inscribed in the boundaries of the bases of the frustum of the cone. The altitude of the frustum of the pyramid coincides with the altitude of the frustum of the cone.

**612. Application of Limits to the Cone.**

Inscribe in a circular cone a pyramid having a base inclosed by a regular polygon; then inscribe a second pyramid whose base is inclosed by a regular polygon having double the number of sides; imagine that this process is continued indefinitely.

Note. — Such pyramids will be called **regular inscribed pyramids**.

It will be assumed evident that the pyramids come nearer and nearer to occupying the same space as the cone. Then:

(a) The **Volume of a Cone** is defined to be the limit of the volume of a regular inscribed pyramid as the number of faces is increased indefinitely.

(b) The **Lateral Area of a Cone** is defined to be the limit of the lateral area of a regular inscribed pyramid as the number of faces is increased indefinitely.

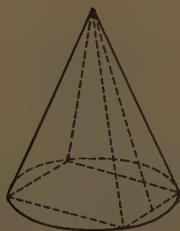
(c) It can be proved that the slant height of a right circular cone is the limit of the slant height of a regular inscribed pyramid as the number of faces is increased indefinitely.

(d) The volume and lateral area of a frustum of a cone are

defined to be the limits of the volume and area respectively of the frustum of a regular inscribed pyramid, having its base inclosed by a regular polygon, as the number of faces is increased indefinitely; also, the slant height of a frustum of a right circular cone can be proved to be the limit of the slant height of the frustum of a regular inscribed pyramid as the number of faces is increased indefinitely.

PROPOSITION VII. THEOREM

613. *The lateral area of a right circular cone is equal to the circumference of its base multiplied by one half its slant height.*



Hypothesis. S = the lateral area, C = the circumference of the base, and L = the slant height of a right circular cone.

Conclusion. $S = \frac{1}{2} CL$.

Proof. 1. Inscribe in the cone a pyramid whose base is inclosed by a regular polygon. Let S' = the lateral area, P' = the perimeter of the base, and L' = the slant height of the pyramid.

2. Then $S' = \frac{1}{2} P'L'$. Why?

3. Let the number of faces of the pyramid increase indefinitely, keeping the pyramid always a regular pyramid. Then

$S' \doteq S$, $P' \doteq C$, and $L' \doteq L$. § 612, b, c

4. $\therefore \frac{1}{2} P'L' \doteq \frac{1}{2} CL$. § 543, c

5. $\therefore S = \frac{1}{2} CL$. § 542, b

614. Cor. If S denotes the lateral area, T the total area, L the slant height, and R the radius of the base of a right circular cone, then

$$S = 2 \pi R \times \frac{1}{2} L = \pi R L.$$

Also,

$$T = \pi R L + \pi R^2 = \pi R(L + R).$$

Ex. 28. Find the lateral area and the total area of a right circular cone, the radius of whose base is 7 in. and whose slant height is 25 in.

Ex. 29. How many square yards of canvas are required for a circus tent having the form of a right circular cylinder, surmounted by a right circular cone, if the diameter of the tent is 100 ft., the height of the vertical wall 15 ft., and the height of the highest point of the tent 50 ft.?

Ex. 30. The diameter of the base of a right circular cone is equal to its altitude. Determine its lateral and total area.

PROPOSITION VIII. THEOREM

615. The volume of a circular cone is equal to the area of its base multiplied by one third the length of its altitude.

Hypothesis. V = the volume, B = the area of the base, and H = the altitude of a circular cone.

Conclusion.

$$V = \frac{1}{3} BH.$$

Proof left to the pupil.

Suggestion. — Model the proof after that in § 601. Use Fig. § 613

616. Cor. If V denotes the volume, H the altitude, and R the radius of the base of a circular cone,

$$V = \frac{1}{3} \pi R^2 H.$$

Ex. 31. Determine the volume of a right circular cone, the radius of whose base is 7 in. and whose slant height is 25 in.

Ex. 32. Determine the volume of the solid generated when a right triangle of base b and altitude h

(a) revolves about its side h ; (b) revolves about its side b .

Ex. 33. Determine the ratio of a circular cone and a circular cylinder having the same base and altitude.

PROPOSITION IX. THEOREM

617. *The lateral area of a frustum of a right circular cone is equal to the sum of the circumferences of its bases, multiplied by one half its slant height.*



Hypothesis. S = the lateral area, C and c the circumferences of the lower and upper bases respectively, and L = the slant height of a frustum of a right circular cone.

Conclusion. $S = \frac{1}{2} L(C + c).$

Proof. 1. Let S' = the lateral area, C' and c' the perimeters of the lower and upper bases, and L' = the slant height of the frustum of a regular pyramid inscribed in the frustum of the cone.

2. $\therefore S' = \frac{1}{2} L'(C' + c').$ Why?

3. Let the number of faces of the frustum of the pyramid be increased indefinitely, keeping the pyramid always a regular pyramid. Then

$S' \doteq S, C' \doteq C, c' \doteq c, L' \doteq L.$ Why?

4. $\therefore \frac{1}{2} L'(C' + c') \doteq \frac{1}{2} L(C + c).$ § 543, c

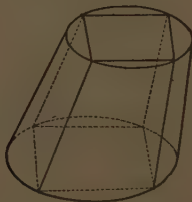
5. $\therefore S = \frac{1}{2} L(C + c).$ § 543, b

618. Cor. 1. *If S denotes the lateral area, L the slant height, and R and r the radii of the bases of a frustum of a right circular cone,* $S = (2\pi R + 2\pi r) \times \frac{1}{2} L = \pi(R + r)L.$

619. Cor. 2. *The lateral area of a frustum of a right circular cone is equal to the circumference of a section midway between the bases, multiplied by the slant height.*

PROPOSITION X. THEOREM

620. *The volume of a frustum of a circular cone is equal to the sum of its bases and the mean proportional between them, multiplied by one third the length of its altitude.*



Hypothesis. V = the volume, B and b = the areas of the lower and upper bases respectively, and H = the length of the altitude of a frustum of a circular cone.

Conclusion. $V = \frac{1}{3} H (B + b + \sqrt{Bb})$.

Proof. 1. Inscribe in the frustum of the cone a frustum of a pyramid having its base inclosed by a regular polygon. Let V' = the volume, and B' and b' = the areas of the lower and upper bases respectively of the frustum of the pyramid.

2. Then $V' = \frac{1}{3} H (B' + b' + \sqrt{B'b'})$. § 572

Complete the proof as in § 601 and § 615.

621. Cor. If V denotes the volume, H the altitude, and R and r the radii of the bases of a frustum of a circular cone, then

$$V = \frac{1}{3} \pi (R^2 + r^2 + Rr) H.$$

Ex. 34. Find the lateral area, the total area, and the volume of a frustum of a cone of revolution, the diameters of whose bases are 16 in. and 6 in., and whose altitude is 12 in.

Ex. 35. Determine the contents in quarts of a water pail having the form of a frustum of a cone of revolution, if the diameters of the bottom and top are 9 in. and 12 in. respectively, and the height of the pail is 14 in. (One quart occupies about $57\frac{1}{4}$ cu. in.)

Note. — Supplementary Exercises 59–71, p. 459, can be studied now.

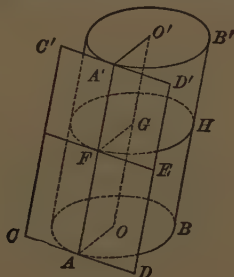
SUPPLEMENTARY TOPICS

A. PLANES TANGENT TO A CYLINDER OR A CONE

622. A plane is tangent to a circular cylinder or to a circular cone when it contains one and only one element of the cylinder or of the cone.

PROPOSITION XI. THEOREM

623. A plane drawn through an element of a circular cylinder and a tangent to the base at its extremity is tangent to the cylinder.



Hypothesis. AA' is an element of the lateral surface of circular cylinder AB' , line CD is tangent to the base AB at A , and plane CD' is drawn through AA' and CD .

Conclusion. CD' is tangent to the cylinder.

Proof. 1. Let E be any point in plane CD' , not in AA' , and draw through E a plane $\parallel$ to the bases, intersecting CD' in line EF and the cylinder in FH .

2. Draw axis OO' ; then OO' is $\parallel AA'$. Ex. 8, p. 392.

3. Let the plane of OO' and AA' intersect the planes of AB and FH in radii OA and GF , respectively.

4. Then, $GF \parallel OA$ and $FE \parallel AD$. Why?

5. $\therefore \angle GFE = \angle OAD$. Why?

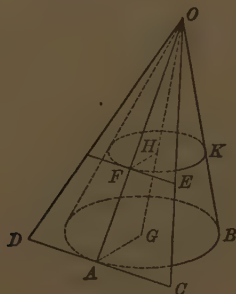
6. $FE \perp GF$, and tangent to $\odot FH$. Prove it.

7. Whence, point E lies outside the cylinder.

8. $\therefore$ all points of CD' , not in AA' , lie outside the cylinder, and CD' is tangent to the cylinder.

PROPOSITION XII. THEOREM

624. *A plane determined by an element of the lateral surface of a circular cone and a tangent to the base at its extremity, is tangent to the cone.*



Hypothesis. OA is an element of the lateral surface of circular cone OAB , line CD is tangent to base AB at A , and plane OCD is drawn through OA and CD .

Conclusion. OCD is tangent to the cone.
(Prove that E lies outside the cone.)

Suggestion. — Model the proof after that of § 623.

B. SIMILAR CYLINDERS AND CONES OF REVOLUTION

625. **Similar cylinders of revolution** are right circular cylinders generated by the revolution of similar rectangles about homologous sides as axes.

Similar cones of revolution are right circular cones generated by the revolution of similar right triangles about homologous sides as axes.

PROPOSITION XIII. THEOREM

626. *The lateral or total areas of two similar cylinders of revolution are to each other as the squares of their altitudes, or as the squares of the radii of their bases; and their volumes are to each other as the cubes of their altitudes, or as the cubes of the radii of their bases.*



Hypothesis. S and s are the lateral areas, T and t are the total areas, V and v are the volumes, H and h are the altitudes, and R and r the radii of the bases, of two similar cylinders of revolution.

Conclusion. $\frac{S}{s} = \frac{T}{t} = \frac{H^2}{h^2} = \frac{R^2}{r^2}$, and $\frac{V}{v} = \frac{H^3}{h^3} = \frac{R^3}{r^3}$.

Proof. 1. Since the generating rectangles are similar,

$$\frac{H}{h} = \frac{R}{r} \quad \text{Why?}$$

$$2. \quad \therefore \frac{H}{h} = \frac{R}{r} = \frac{H+R}{h+r} \quad \S\ 255, \S\ 253$$

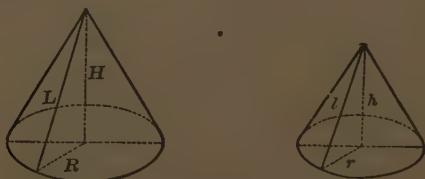
$$3. \quad \frac{S}{s} = \frac{2\pi RH}{2\pi rh} = \frac{R}{r} \times \frac{R}{r} = \frac{R^2}{r^2} = \frac{H^2}{h^2} \quad \text{Why?}$$

$$4. \quad \frac{T}{t} = \frac{2\pi R(H+R)}{2\pi r(h+r)} = \frac{R}{r} \times \frac{R}{r} = \frac{R^2}{r^2} = \frac{H^2}{h^2} \quad \text{Why?}$$

$$5. \quad \frac{V}{v} = \frac{\pi R^2 H}{\pi r^2 h} = \frac{R^2}{r^2} \times \frac{R}{r} = \frac{R^3}{r^3} = \frac{H^3}{h^3} \quad \text{Why?}$$

PROPOSITION XIV. THEOREM

627. *The lateral or total areas of two similar cones of revolution are to each other as the squares of their slant heights, or as the squares of their altitudes, or as the squares of the radii of their bases; and their volumes are to each other as the cubes of their slant heights, or as the cubes of their altitudes, or as the cubes of the radii of their bases.*



Hypothesis. S and s are the lateral areas, T and t the total areas, V and v are the volumes, L and l are the slant heights, H and h are the altitudes, and R and r the radii of the bases, of two similar cones of revolution. (§ 625.)

Conclusion. $\frac{S}{s} = \frac{T}{t} = \frac{L^2}{l^2} = \frac{H^2}{h^2} = \frac{R^2}{r^2}$, and $\frac{V}{v} = \frac{L^3}{l^3} = \frac{H^3}{h^3} = \frac{R^3}{r^3}$.

The proof is left to the pupil; model it after that of § 626.

Ex. 36. At what distance from the vertex of a right circular cone with altitude H must a plane parallel to the base be passed so that the lateral area will be bisected?

BOOK IX

THE SPHERE

628. A **Spherical Surface** is a closed surface all points of which are equidistant from a point within, called the **Center**.

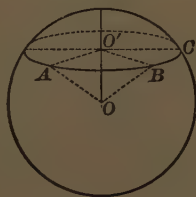
629. A **Sphere** is the solid bounded by a spherical surface.

630. A **Radius** of a sphere, or of its surface, is the straight line drawn from its center to any point of its surface.

A **Diameter** of a sphere, or of its surface, is the straight line drawn through the center having its extremities in the surface.

PROPOSITION I. THEOREM

631. *The intersection of a spherical surface and a plane is a circle.*



Hypothesis. ABC is the intersection of the spherical surface whose center is O and a plane.

Conclusion. Curve ABC is a $\odot$

Proof. 1. Draw $OO' \perp$ plane of ABC .

2. Let A and B be any two points in curve ABC . Draw OA , OB , $O'A$, and $O'B$.

Suggestion. — Now prove that $O'A = O'B$, and ABC is a $\odot$.

632. A **Great Circle** of a sphere is the intersection of its surface and a plane passing through its center; as $\odot ABC$.

A **Small Circle** of a sphere is the intersection of its surface and a plane which does not pass through its center.

The **Axis** of a circle of a sphere is the diameter of the sphere which is perpendicular to the plane of the circle; as axis POP' .

The **Poles** of a **Circle** of a sphere are the extremities of the axis of the circle.

633. Cor. 1. *The axis of a circle of a sphere passes through the center of the circle.*

634. Cor. 2. *All great circles of a sphere are equal.*

635. Cor. 3. *Every great circle bisects the sphere and its surface.*

For if the portions of the sphere formed by the plane of the great circle be separated, and placed so that their plane surfaces coincide, the spherical surfaces falling on the same side of this plane, the two spherical surfaces will coincide throughout; for all points of either surface are equally distant from the center.

636. Cor. 4. *Any two great circles bisect each other.*

For the intersection of their planes is a diameter of the sphere, and therefore a diameter of each circle.

637. Cor. 5. *Between any two points on the surface of a sphere, not the extremities of a diameter, one and only one arc of a great circle, less than a semicircle, can be drawn.*

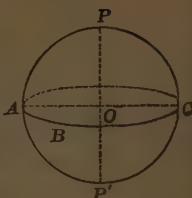
For the two points, with the center of the sphere, determine a plane which intersects the surface of the sphere in the required arc.

638. *Two spheres are equal when their radii are equal.*

All radii and diameters of the same sphere or equal spheres are equal.

639. A spherical surface may be generated by the revolution of a semicircle about its diameter as an axis.

For all points of such a surface are equally distant from the center of the circle.



640. Through two points of a spherical surface, an infinity of small circles of the sphere can be drawn. Why?

Through three points of a spherical surface, one and only one small circle of the sphere can be drawn. Why?

Ex. 1. Prove that a great circle of a sphere which passes through one pole of a circle must pass through the other pole also.

Ex. 2. How many great circles can be passed through two points which are the extremities of a diameter of a sphere?

Ex. 3. What kind of circles of the earth are the parallels of latitude?

Ex. 4. What kind of circles of the earth are the meridians?

Ex. 5. What kind of circle of the earth is the equator?

Ex. 6. Speaking strictly, is it accurate to speak of the North Pole of the earth? Of what circle or circles is it a pole?

Ex. 7. Into how many parts do two great circles of a sphere divide the surface of the sphere?

Ex. 8. Into how many parts do three great circles of a sphere divide the surface of the sphere, if they do not all have a common diameter?

Ex. 9. Prove that all circles of a sphere made by parallel planes have the same axis and the same poles.

Ex. 10. Given a point of a spherical surface. Prove that it is the pole of one and only one great circle.

Ex. 11. Prove that circles of a sphere made by planes equidistant from the center of the sphere are equal.

Suggestion.—Use the Pythagorean Theorem. (§ 291.)

Ex. 12. State and prove the converse of Ex. 11.

Ex. 13. Prove that circles of a sphere made by planes unequally distant from the center of the sphere are unequal, the more remote being the smaller.

Ex. 14. State and prove the converse of Ex. 13.

Ex. 15. In how many points can two straight lines intersect?

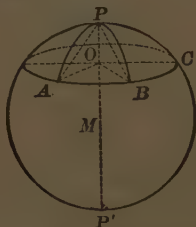
In how many points on one hemisphere can two great circles intersect?

641. The great circles of a sphere in *spherical geometry* correspond to the straight lines of a plane in *plane geometry*. § 637 and Ex. 15 are two instances pointing to this similarity of great circles and straight lines; others will appear in the remaining paragraphs of Book IX.

642. It can be proved that the length of the arc of the great circle, less than a semicircle, between two points of a spherical surface is less than the length of any other curved line on the surface between the two points. Consequently, the *distance between two points on the surface of a sphere*, measured on the surface, is defined to be the arc of the great circle, less than a semicircle, drawn between them.

PROPOSITION II. THEOREM

643. *All points in a circle of a sphere are equidistant from each of its poles.*



Hypothesis. P and P' are the poles of $\odot ABC$ of sphere M .

Conclusion. All points of $\odot ABC$ are equidistant from P , and also from P' .

Proof. 1. Let A and B be any two points of $\odot ABC$, and draw great circle arcs PA and PB . Draw axis PMP' , intersecting the plane of ABC at O . Draw OA , OB , PA , and PB .

2. $\therefore PA = PB$. Prove it.

3. $\therefore \widehat{PA} = \widehat{PB}$. Prove it.

4. Since A and B are any two points of $\odot ABC$, $\therefore$ all points of $\odot ABC$ are equidistant from P .

5. Similarly all points of $\odot ABC$ are equidistant from P' .

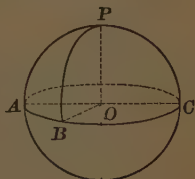
644. The **Polar Distance** of a circle of a sphere is the distance from the nearer of its poles to the circle, or from either pole if they are equally near.

Thus, in the figure of Proposition II, the polar distance of $\odot ABC$ is arc PA .

645. Cor. *All points of a great circle of a sphere are at a quadrant's distance from either of its poles.*

Note. — The term *quadrant* in Spherical Geometry, usually signifies a quadrant of a great circle.

Hypothesis. P is a pole of great circle ABC of sphere APC ; B is any point in $\odot ABC$, and PB is an arc of a great $\odot$.



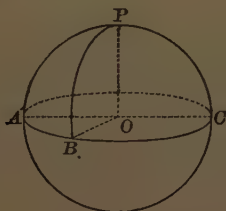
Conclusion. Arc PB = a quadrant.

Suggestion. — Draw radii OA , OB , and OP .

Note. — An arc of a circle may be drawn on the surface of a sphere by placing one foot of the compasses at the nearer pole of the circle, the distance between the feet being equal to the chord of the polar distance.

PROPOSITION III. THEOREM

646. *A point on the surface of a sphere at a quadrant's distance from each of two points, not the extremities of a diameter of the sphere, is a pole of the great circle through those points.*



Hypothesis. P is on the surface of the sphere whose center is O . AB is an arc of great $\odot ABC$, not a semicircle. $\widehat{PA}$ and $\widehat{PB}$ are quadrants.

Conclusion. P is a pole of $\widehat{AB}$.

Suggestions. — 1. Recall the definition of "pole of a $\odot$."

2. Draw PO , AO , OB , and prove $PO \perp$ plane ABC .

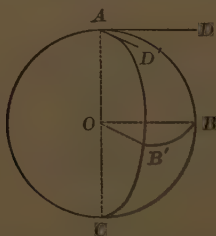
Ex. 16. If a point lies at a quadrant's distance from the ends of a diameter of a sphere, is it necessarily a pole of the great circle through those points?

647. The angle between two intersecting curves is the angle formed by the tangents to the curves at the point of intersection.

A **Spherical Angle** is the angle between two intersecting arcs of great circles.

PROPOSITION IV. THEOREM

648. A spherical angle is measured by an arc of a great circle having its vertex as a pole, included between its sides extended if necessary.



Hypothesis. ABC and $AB'C$ are arcs of great $\odot$ on the surface of the sphere whose center is O ; lines AD and AD' are tangent to ABC and $AB'C$, respectively, and BB' is an arc of a great $\odot$ having A as a pole, included between arcs ABC and $AB'C$.

Conclusion. $\angle BAB'$ is measured by arc BB' .

Proof. 1. Draw diameter AOC and radii OB and OB' .

- | | | |
|----------|---|-------|
| 2. | Arcs AB and AB' are quadrants. | Why? |
| 3. | $\therefore \angle AOB$ and $\angle AOB'$ are rt. $\angle$ s. | Why? |
| 4. | $OB \parallel AD$ and $OB' \parallel AD'$. | Why? |
| 5. | $\therefore \angle DAD' = \angle BOB'$. | § 481 |
| 6. But | $\angle BOB'$ is measured by arc BB' . | Why? |
| 7. Then, | $\angle DAD'$ is measured by arc BB' . | Why? |
| 8. | $\therefore \angle BAB'$ is measured by arc BB' . | § 647 |

649. Cor. 1. *The angle between two arcs of great circles is equal to the dihedral angle formed by their planes.*

650. Cor. 2. *An arc of a great circle drawn to another great circle from the latter's pole is perpendicular to that great circle.*

Suggestions. — 1. What $\angle$ does OA form with plane BOB' ?

2. What kind of $\angle$ is dihedral $\angle AOB'B$? (§ 495.)

Ex. 17. If a spherical blackboard can be had, construct a spherical angle and measure it.

Ex. 18. The chord of the polar distance of a circle of a sphere is 6. If the radius of the sphere is 5, what is the radius of the circle?

Ex. 19. What is the locus of points in space at a given distance d from a fixed point, and equidistant from two given points?

SPHERICAL POLYGONS

Note. — Recall at this point the definition of a polygon in plane geometry as a closed broken line lying in a plane. (§ 125.)

Recall also § 641, calling attention to the similarity in the rôles of the straight line in plane geometry and the great circle in solid geometry.

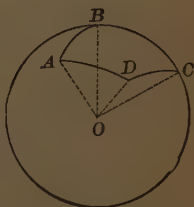
651. A **Spherical Polygon** is a closed line on the surface of a sphere consisting of arcs of three or more great circles; as polygon $ABCD$.

Just as in plane geometry we considered only convex polygons, so we shall consider only convex spherical polygons. (See § 126.)

It will be assumed as evident that a simple spherical polygon incloses a portion of the surface of the sphere. The bounding arcs are the **Sides** of the polygon; they are usually measured in arc-degrees. (§ 214.)

The angle formed by two consecutive sides of a polygon is an **Angle** of the spherical polygon, and its vertex is a **Vertex** of the polygon.

A **Diagonal** of a spherical polygon is the arc of the great circle joining two non-consecutive vertices of the polygon, and lying within the polygon.



652. A **Spherical Triangle** is a spherical polygon having three sides. A spherical triangle is **Isosceles** when it has two equal sides; it is **Equilateral** when it has three equal sides; it is **Right-angled** when one of its angles is a right angle.

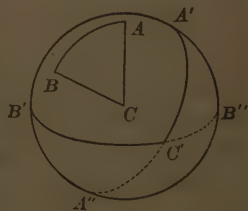
653. The planes of the sides of a spherical polygon form a polyedral angle, whose vertex is the center of the sphere, and whose face angles are measured by the sides of the spherical polygon.

Thus, in the figure of § 651, the planes of the sides of the spherical polygon form a polyedral, $\angle O-ABCD$, whose face angles AOB , BOC , etc., are measured by arcs AB , BC , etc., respectively.

Ex. 20. Prove that the angles of a spherical polygon have the same measures as the diedral angles of the corresponding polyedral angle.

654. If great circles be drawn with the vertices of a spherical triangle as poles, they divide the surface of the sphere into eight parts whose boundaries are tri-angles.

Thus, if circle $B'C'B''$ be drawn with vertex A of spherical $\triangle ABC$ as a pole, circle $A'C'A''$ with B as a pole, and circle $A'B'A''B'$ with C as a pole, the surface of the sphere is divided into eight spherical $\triangle$; namely, $A'B'C'$, $A'B''C'$, $A''B'C'$, and $A''B''C'$ on the hemisphere represented in the figure, and four others on the opposite hemisphere.



Of these eight spherical $\triangle$, one is called the **Polar Triangle** of ABC , and is determined as follows:

Of the intersections, A' and A'' of circles drawn with B and C as poles, that which is nearer to A , i.e. A' , is a vertex of the polar triangle; and similarly for the other intersections.

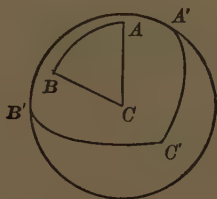
Thus, $A'B'C'$ is the polar $\triangle$ of ABC .

Ex. 21. The polar distance of a circle of a sphere is 60° . If the diameter of the circle is 6, find the diameter of the sphere, and the distance of the circle from its center.

Suggestion. — Represent the radius of the sphere by $2x$. (§ 288.)

PROPOSITION V. THEOREM

655. *If one spherical triangle is the polar triangle of another, then the second spherical triangle is the polar triangle of the first.*



Hypothesis. $A'B'C'$ is the polar Δ of the spherical ΔABC ; A , B , and C are the poles of arcs $B'C'$, $C'A'$, and $A'B'$, respectively.

Conclusion. ABC is the polar Δ of spherical $\Delta A'B'C'$.

Proof. 1. A' is at a quadrant's distance from B . § 645
(Since B is the pole of arc $A'C'$.)

2. A' is at a quadrant's distance from C . Why?

3. $\therefore A'$ is a pole of the great $\odot$ arc BC . Why?

4. Similarly B' and C' are the poles of $\widehat{AC}$ and $\widehat{AB}$, respectively. Prove it.

5. $\therefore \Delta ABC$ is the polar Δ of $\Delta A'B'C'$.

(For of the two intersections of the great $\odot$ having B' and C' , respectively, as poles, A is nearer to A' ; similarly for B and C . § 654)

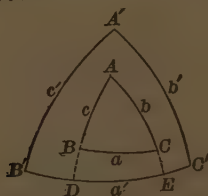
Note. — Two spherical triangles, each of which is the polar triangle of the other, are called polar triangles.

Ex. 22. How many degrees are there in the polar distance of a circle whose plane is $5\sqrt{2}$ units from the center of the sphere, the diameter of the sphere being 20 units?

Suggestion. — The radius of the $\odot$ is a leg of a rt. Δ , whose hypotenuse is the radius of the sphere, and whose other leg is the distance from its center to the plane of the $\odot$.

PROPOSITION VI. THEOREM

656. *In two polar triangles, each angle of one has the same measure as the supplement of that side of the other of which it is the pole.*



Hypothesis. $\triangle ABC$ and $A'B'C'$ are polar $\triangle$ s, point A being the pole of $\widehat{B'C'}$, etc.

Let a, a' , etc. be the measures in degrees of $\widehat{BC}, \widehat{B'C'}$, etc., respectively; let A, A' , etc. be the measures in degrees of $\angle A, \angle A'$, etc.

Conclusion. $A = 180 - a'$; $B = 180 - b'$; $C = 180 - c'$.
 $A' = 180 - a$; $B' = 180 - b$; $C' = 180 - c$.

Proof. 1. Extend $\widehat{AB}$ and $\widehat{AC}$ to meet $\widehat{B'C'}$ at D and E , respectively.

2. Since B' is the pole of $\widehat{AE}$, $\widehat{B'E} = 90^\circ$. Why?

3. Similarly, $\widehat{C'D} = 90^\circ$.

4. $\therefore \widehat{B'E} + \widehat{C'D} = 180^\circ$.

5. $\therefore \widehat{B'D} + \widehat{DE} + \widehat{C'D} = 180^\circ$, or $\widehat{DE} + \widehat{B'C'} = 180^\circ$.

6. But $\widehat{DE}$ is the measure of $\angle A$. § 648

7. $\therefore A + a' = 180$, or
 $A = 180 - a'$.

8. Similarly for each of the other angles of either triangle.

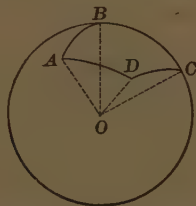
Ex. 23. Prove on the figure for § 656, that $A' = 180 - a$.

Ex. 24. If the sides of a spherical triangle are 77° , 123° , and 95° , how many degrees are there in each angle of its polar triangle?

Ex. 25. If the angles of a spherical triangle are 86° , 131° , and 68° , how many degrees are there in each side of its polar triangle?

PROPOSITION VII. THEOREM

657. *The sum of the sides of a convex spherical polygon is less than 360° .*



Hypothesis. $ABCD$ is a convex spherical polygon.

Conclusion. $\widehat{AB} + \widehat{BC} + \widehat{CD} + \widehat{DA} < 360^\circ$.

Proof. 1. Let polyedral angle $O-ABCD$ be the polyedral angle which corresponds to spherical polygon $ABCD$.

2. Then the measure of $\widehat{AB}$ equals the measure of central angle AOB . Why?

3. Similarly for arcs BC , CD , and DA .

4. But $\angle AOB + \angle BOC + \angle COD + \angle DOA < 360^\circ$.

§ 514

5. $\therefore \widehat{AB} + \widehat{BC} + \widehat{CD} + \widehat{DA} < 360^\circ$.

Note 1. — The pupil should recall at this point that one *arc-degree* is $\frac{1}{360}$ of a circle. Since arcs AB , BC , CD , and DA are arcs of great circles, Proposition VII means that the sum of the sides of any spherical polygon is less than 360 arc-degrees of a great circle — i. e. *is less than a great circle*.

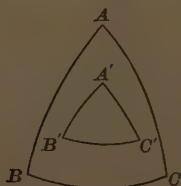
Note 2. — Proposition VII is one of many theorems about spherical polygons which can be formulated from corresponding theorems about polyedral angles by replacing in the latter the words “face angle” and “diedral angle” by “side” and “angle” respectively.

Ex. 26. If two sides of a spherical triangle measure 80° and 70° respectively, between what two values must the remaining side lie?

Ex 27. What is the greatest possible length for the sum of the sides of a convex spherical polygon on a sphere of radius 12 inches?

PROPOSITION VIII. THEOREM

658. *The sum of the angles of a spherical triangle is greater than 180° and less than 540° .*



Hypothesis. A , B , and C are the measures in degrees of the $\angle$ s of spherical $\triangle ABC$.

Conclusion. $A + B + C > 180^\circ$ and $< 540^\circ$.

Proof. 1. Let $\triangle A'B'C'$ be the polar triangle of spherical $\triangle ABC$, A being the pole of $\widehat{B'C'}$, B of $\widehat{A'C'}$, and C of $\widehat{A'B'}$.

Let the measures in degrees of $\widehat{B'C'}$, $\widehat{C'A'}$, and $\widehat{A'B'}$ be a' , b' , and c' , respectively.

2. $\therefore A = 180 - a'$, $B = 180 - b'$, and $C = 180 - c'$. Why?

3. $\therefore A + B + C = 540 - (a' + b' + c')$. Why?

4. $\therefore A + B + C < 540^\circ$.

5. But $a' + b' + c' < 360^\circ$. § 657

6. $\therefore A + B + C > 180^\circ$. Ax. 20, § 158

659. Cor. *The sum of the angles of a spherical polygon of n sides is greater than $(n - 2) \times 180^\circ$.*

Consider $ABCD$ a spherical quadrilateral.

Draw great $\odot$ arc BD , dividing the quadrilateral into two spherical $\triangle$ s, ABD and BDC . The sum of the angles of the triangles equals the sum of the angles of the quadrilateral. In each triangle the sum of the angles is $> 180^\circ$. Hence for the quadrilateral, the sum of the angles is greater than $2 \times 180^\circ$; that is, the sum $> (4 - 2) \times 180^\circ$.

In like manner, if there are n sides, the polygon can be divided into $(n - 2)$ spherical triangles, in each of which the sum of the angles is greater than 180° . Therefore, in the polygon the sum of the angles is greater than $(n - 2) \times 180^\circ$.

660. The **Spherical Excess** of a spherical triangle, measured in degrees, is the difference between the sum of its angles and 180° .

The **Spherical Excess** of a spherical polygon of n sides, measured in degrees, is the difference between the sum of its angles and $(n - 2) \times 180^\circ$.

Note.— In each case, the spherical excess is the amount by which the sum of the angles of the spherical polygon exceeds the sum of the angles of a plane polygon of the same number of sides.

Ex. 28. Prove that a spherical triangle may have one, two, or three right angles, or one, two, or three obtuse angles.

661. A spherical triangle having two right angles is called a **Bi-rectangular Triangle**, and one having three right angles a **Tri-rectangular Triangle**.

Ex. 29. Prove that the sum of the angles of a spherical hexagon is greater than 8, and less than 12, right angles.

Ex. 30. What is the spherical excess of a triangle whose angles are 100° , 95° , and 65° , respectively?

Ex. 31. What is the spherical excess of a tri-rectangular triangle?

Ex. 32. Prove that the spherical excess of a bi-rectangular triangle is the measure of the remaining angle of the triangle. (§ 648.)

Ex. 33. What is the spherical excess of a triangle if the sides of its polar triangle measure 80° , 85° , and 95° ?

Ex. 34. What relation exists between a tri-rectangular spherical triangle and its polar?

Ex. 35. Prove that the sides opposite the equal angles of a bi-rectangular triangle are quadrants.

Suggestion.— Recall § 499 and the definition of “pole.”

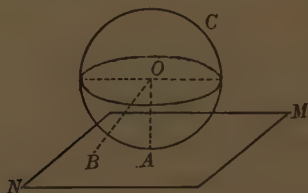
Ex. 36. Prove that each side of a tri-rectangular triangle is a quadrant.

Ex. 37. Prove that in a bi-rectangular spherical triangle, the third angle has the same measure as the side opposite it.

662. If a plane or a line has only one point in common with the surface of a sphere, it is said to be **Tangent to the Sphere**. The sphere is said to be tangent to the plane or line. The point common to the plane or line and the spherical surface is called the **Point of Contact or Tangency**.

PROPOSITION IX. THEOREM

663. *A plane perpendicular to a radius of a sphere at its outer extremity is tangent to the sphere.*



Hypothesis. Plane $MN \perp$ radius OA at A .

Conclusion. Plane MN is tangent to the sphere.

Suggestion. — Let B be any point of MN except A .

Prove that B lies outside the sphere.

664. Cor. *A plane tangent to a sphere is perpendicular to the radius drawn to the point of contact.* (Fig. of Prop. IX.)

665. Two Spheres are Tangent to each other when each is tangent to the same plane at the same point.

Ex. 38. Prove that a straight line perpendicular to a radius of a sphere at its outer extremity is tangent to the sphere.

Ex. 39. How many straight lines can be tangent to a sphere at a point of the sphere?

Ex. 40. If two straight lines are tangent to a sphere at the same point, their plane is tangent to the sphere.

Ex. 41. Prove that all lines tangent to a sphere at a point of the sphere lie in the plane tangent to the sphere at that point.

Ex. 42. How many straight lines can be tangent to a sphere from a point outside the sphere? Compare the lengths of these tangents.

Ex. 43. Prove that the points of contact of all lines tangent to a sphere from an exterior point lie in a circle.

Ex. 44. Is the circle in Ex. 43 a great circle or a small circle?

Ex. 45. If two spheres are tangent to the same plane at the same point, the straight line joining their centers passes through the point of contact.

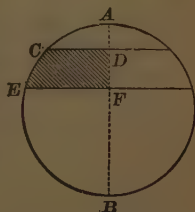
MEASUREMENT OF SPHERICAL POLYGONS

666. A **Zone** is the portion of a spherical surface included between two parallel planes.

The circles which bound the zone are its *bases*, and the distance between their planes is its *altitude*.

A *zone of one base* is a zone lying between one plane and a parallel plane tangent to the sphere.

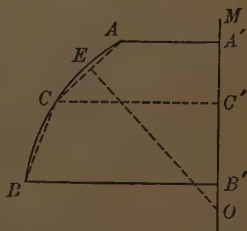
667. If semicircle $ACEB$ be revolved about diameter AB as an axis, and CD and EF are lines $\perp AB$, then arc CE generates a zone whose altitude is DF , and arc AC a zone of one base whose altitude is AD .



668. Application of Limits to Zones.

Let O be the center of $\widehat{ACB}$, and OM be any diameter of the circle. Let AA' and BB' be $\perp OM$. Let C bisect arc AB , and draw broken line ACB . If arc AB is revolved about OM as axis, it generates a zone, and broken lines AC and CB generate curved surfaces.

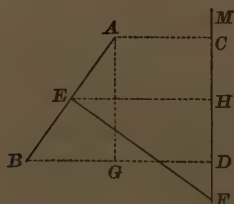
It will be assumed as evident that the sum of the areas of the surfaces generated by AC and CB is less than the area of the zone generated by $\widehat{ACB}$.



Assume arcs AC and CB to be bisected at X and Y , and imagine the broken line $AXCYB$. It will be assumed as evident that the area of the surface generated by line $AXCYB$ is greater than that generated by ACB but is still less than that of the zone generated by $\widehat{AB}$. If the process of subdividing arc AB by successively halving the subdivisions of arc AB be continued indefinitely, it will be assumed evident that the surface generated by the resulting broken line approaches as limit the zone AB ; also, as the chords like AC decrease indefinitely in length, their distance from center O increases and approaches the radius of arc AB as limit.

PROPOSITION X. THEOREM

669. *The area of the surface generated by the revolution of a straight line about a straight line in its plane, not parallel to and not intersecting it, as an axis, is equal to its projection on the axis, multiplied by the circumference of a circle, whose radius is the perpendicular erected at the mid-point of the line and terminating in the axis.*



Hypothesis. Straight line AB is revolved about straight line FM in its plane, not $\parallel$ to and not intersecting it, as an axis; lines AC and $BD \perp FM$, and EF is the $\perp$ erected at the mid-point of AB terminating in FM .

Conclusion. Area $AB^* = CD \times 2\pi EF$.

Proof. 1. Draw line $AG \perp BD$, and line $EH \perp CD$.

2. The surface generated by AB is the lateral surface of a frustum of a cone of revolution, whose bases are generated by AC and BD .

3. $\therefore \text{area } AB = AB \times 2\pi EH$ § 619

4. $\triangle ABG$ and EFH are similar. Prove it.

5. $\therefore \frac{AB}{AG} = \frac{EF}{EH}$ Why?

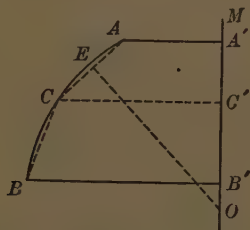
6. $\therefore AB \times EH = AG \times EF$ Why?
 $= CD \times EF$ Why?

7. Substituting in Step 3,
 $\text{area } AB = CD \times 2\pi EF$.

*The expression "area AB " is used to denote the area of the surface generated by AB .

PROPOSITION XI. THEOREM

670. *The area of a zone is equal to its altitude multiplied by the circumference of a great circle.*



Hypothesis. $\widehat{AB}$ is revolved about diameter OM as axis; AA' and $BB' \perp OM$; R is the radius of $\widehat{AB}$.

Conclusion. Area of zone generated by $\widehat{AB} = A'B' \times 2\pi R$.

Proof. 1. Bisect $\widehat{AB}$ at C ; draw AC and CB ; also draw $CC' \perp OM$ and $OE \perp AC$.

2. Revolve the figure about OM as axis.

3. $\therefore$ area $AC = A'C' \times 2\pi OE$. § 669

area $CB = C'B' \times 2\pi OE$. Why?

4. Adding, the surface generated by ACB

$$= (A'C' + C'B') \times 2\pi OE.$$

5. Continue to bisect the subdivisions of $\widehat{AB}$, indefinitely.

6. Then, the area of the surface generated by revolving the inscribed broken line approaches as limit the area of the zone generated by $\widehat{AB}$, and

$$OE \doteq R. \quad \S 668$$

7. $\therefore$ area of zone $= A'B' \times 2\pi R$. § 403

Note. — The proof of § 670 holds for any zone which lies entirely on the surface of a hemisphere; for, in that case, no chord is $\parallel OM$, and § 669 is applicable.

Since a zone which does not lie entirely on the surface of a hemisphere may be considered as the sum of two zones, each of which does lie entirely on the surface of a hemisphere, the theorem of § 670 is true for any zone

671. Cor. 1. If Z denotes the area of a zone, h its altitude, and R the radius of the sphere,

$$Z = 2 \pi R h.$$

672. Cor. 2. The area of a spherical surface equals the square of its radius multiplied by 4π .

Proof. A spherical surface may be regarded as a zone whose altitude is a diameter of the sphere. Letting S represent the area of the spherical surface,

$$S = 2 \pi R \cdot 2 R = 4 \pi R^2.$$

673. Cor. 3. The area of the surface of a sphere equals the area of four great circles of the sphere.

674. Cor. 4. The areas of two spherical surfaces have the same ratio as the squares of their radii or the squares of their diameters.

675. A **Lune** is the portion of a spherical surface bounded by two semicircles of great circles; as $ACBD$.

The **Angle of a Lune** is the angle between its bounding arcs.

It is evident that two lunes on the same sphere or equal spheres are congruent if their angles are equal.

676. It is evident that lunes on the same sphere or on equal spheres may be added by placing them so that their angles become adjacent angles; thus lune $ACBE$ + lune $AEBF$ = lune $ACBF$.

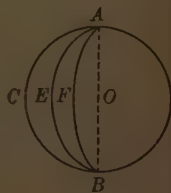
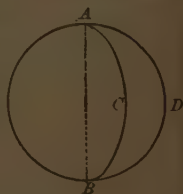
When two lunes are added, the angle of the sum equals the sum of the angles of the given lunes.

If L_X is used to denote the lune whose angle is $\angle X$, then

$$L_X + L_Y = L_{(X+Y)};$$

i.e. lune of $\angle X$ + lune of $\angle Y$ = lune of $\angle (X + Y)$.

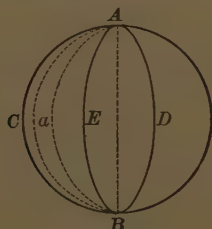
Note.—It is very important that the symbol L_X be understood and remembered.



PROPOSITION XII. THEOREM

677. *Two lunes on the same sphere or equal spheres have the same ratio as their angles.*

CASE I. *When the angles are commensurable.*



Hypothesis. $ACBD$ and $ACBE$ are lunes on sphere AB , having their $\angle CAD$ and CAE commensurable.

Conclusion.
$$\frac{ACBD}{ACBE} = \frac{\angle CAD}{\angle CAE}.$$

Proof. 1. Let $\angle CAa$ be a common measure of $\angle CAD$ and CAE , and let it be contained 5 times in $\angle CAD$, and 3 times in $\angle CAE$.

$$2. \quad \therefore \frac{\angle CAD}{\angle CAE} = \frac{5}{3}. \quad (1)$$

3. Extending the arcs of division of $\angle CAD$ to B , lune $ACBD$ will be divided into 5 parts, and lune $ACBE$ into 3 parts, all of which parts will be equal. Why?

$$4. \quad \therefore \frac{ACBD}{ACBE} = \frac{5}{3}. \quad (2)$$

$$5. \text{ From (1) and (2), } \frac{ACBD}{ACBE} = \frac{\angle CAD}{\angle CAE}. \quad \text{Why?}$$

Note. — The theorem may be proved in a similar manner when the given lunes are on equal spheres.

CASE II. *When the angles are incommensurable.*

Suggestion. — Model the proof after that in § 544.

678. The surface of a hemisphere may be regarded as a lune of angle 180° and the surface of the sphere, a lune of angle 360° .

679. Cor. 1. *The surface of a lune is to the surface of the sphere as the measure of its angle in degrees is to 360.*

680. Cor. 2. *If the radius of the sphere is R , and the angle of the lune, measured in degrees, is A , and the area of the lune is denoted by L_A , then*

$$L_A = \frac{A}{360} \times 4\pi R^2 = \frac{\pi R^2 A}{90}.$$

Ex. 46. Prove that the areas of two zones on the same sphere, or equal spheres, are to each other as their altitudes.

Ex. 47. Determine the area of a zone whose altitude is 13, if the radius of the sphere is 16.

Ex. 48. Prove that the area of a zone of one base is equal to the area of the circle whose radius is the chord of its generating arc. (§ 288.)

Ex. 49. Determine the area of the surface of a sphere whose radius is 12.

Ex. 50. If the radius of a sphere is R , what is the area of a zone of one base, whose generating arc is 45° ?

Ex. 51. Find the radius of a sphere whose surface is equivalent to the entire surface of a cylinder of revolution, whose altitude is $10\frac{1}{2}$, and radius of base 3.

Ex. 52. What is the area of a lune whose angle is 40° on the surface of a sphere whose radius is 15 in.?

Ex. 53. What part of the surface of the earth is included between the 30th and 35th meridians?

Ex. 54. The area of a lune is $28\frac{1}{2}$. If the area of the surface of the sphere is 120, what is the angle of the lune?

Ex. 55. Prove that the surface of a sphere is equal to two thirds the entire surface of the right circular cylinder circumscribed about it.

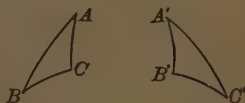
Ex. 56. Compare the surface of a sphere with the lateral surface of the right circular cylinder circumscribed about the sphere.

Ex. 57. What part of the surface of the earth lies in the zone between the equator and the parallel of latitude 30° N.?

Ex. 58. What zones of the earth are zones of one base?

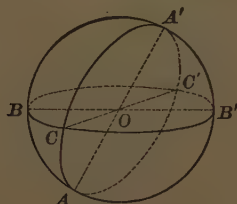
681. Two spherical polygons, on the same or equal spheres, are **Symmetrical** when the sides and angles of one are equal, respectively, to the sides and angles of the other, if the equal parts occur in opposite orders.

Thus, if spherical $\triangle ABC$ and $A'B'C'$, on the same or equal spheres, have sides AB , BC , and CA equal, respectively, to sides $A'B'$, $B'C'$, and $C'A'$, and $\angle A$, B , and C to $\angle A'$, B' , and C' , and the equal parts occur in the opposite orders the $\triangle$ are symmetrical.



PROPOSITION XIII. THEOREM

682. *The spherical triangles corresponding to a pair of vertical triedral angles are symmetrical.*



Hypothesis. AOA' , BOB' , and COC' are diameters of the sphere with center O ; the planes determined by them intersect the spherical surface in $\odot ABA'B'$, $ACA'C'$, and $BCB'C'$.

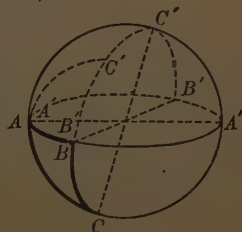
Conclusion. Spherical $\triangle ABC$ and $A'B'C'$ are symmetrical.

Suggestions. — 1. Prove $\widehat{A'B'} = \widehat{AB}$; $\widehat{B'C'} = \widehat{BC}$, etc.

2. Prove $\angle BCA = \angle B'C'A'$; $\angle BAC = \angle B'A'C'$; etc.

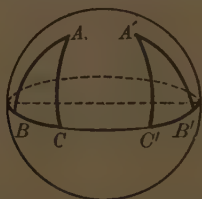
3. Prove that the parts of $\triangle ABC$ occur in opposite order to those of $\triangle A'B'C'$.

The adjoining figure will aid in doing this. $A'B'C'$ has been slid around the sphere until it occupies the position indicated in this figure. Determine the direction from A to B to C ; and also the direction from A' to B' to C' .



PROPOSITION XIV. THEOREM

683. *Two symmetrical spherical triangles, of which one is isosceles, are congruent and hence equal.*



Hypothesis. $\triangle ABC$ is symmetrical to $\triangle A'B'C'$; that is, $\widehat{AB} = \widehat{A'B'}$, $\widehat{AC} = \widehat{A'C'}$, $\widehat{BC} = \widehat{B'C'}$, $\angle B = \angle B'$, $\angle C = \angle C'$, $\angle A = \angle A'$, with the equal parts arranged in opposite orders in the triangles; also $\widehat{AB} = \widehat{AC}$.

Conclusion. $\triangle ABC \cong \triangle A'B'C'$.

Proof. 1. Since $\widehat{AB} = \widehat{A'B'}$, and $\widehat{AB} = \widehat{AC}$,
 $\therefore \widehat{A'B'} = \widehat{AC}$.

2. In like manner $\widehat{A'C'} = \widehat{AB}$.

Complete the proof by superposing $\triangle A'B'C'$ on $\triangle ABC$, making $\widehat{A'C'}$ coincide with $\widehat{AB}$, with point A' on point A .

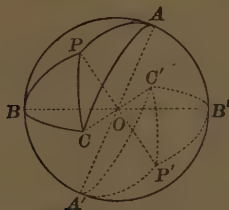
684. Cor. *If one of two symmetrical spherical triangles is isosceles, the other is also.*

PROPOSITION XV. THEOREM

685. *Two spherical triangles corresponding to a pair of vertical triedral angles are equal.*

Hypothesis. AOA' , BOB' , and COC' are diameters of sphere O ; also, the planes determined by them intersect the surface in arcs AB , BC , AC , $A'B'$, $B'C'$, and $A'C'$. (Fig. p. 431.)

Conclusion. Area of $\triangle ABC$ = area of triangle $A'B'C'$.



Proof. 1. Let P be the pole of the small circle passing through points A , B , and C ; draw arcs of great circles PA , PB , and PC .

2. $\therefore \widehat{PA} = \widehat{PB} = \widehat{PC}$. § 643

3. Draw PP' , a diameter of the sphere, and $P'A'$, $P'B'$, and $P'C'$ arcs of great $\odot$; then spherical $\triangle PAB$ and $P'A'B'$ are symmetrical. § 682

4. But spherical $\triangle PAB$ is isosceles.

5. $\therefore \triangle PAB = \triangle P'A'B'$. Why?

6. In like manner,

$$\triangle PBC = \triangle P'B'C' \text{ and } \triangle PCA = \triangle P'C'A'.$$

7. Then the sum of the areas of $\triangle PAB$, PBC , PAC equals the sum of the areas of $\triangle P'A'B'$, $P'B'C'$, and $P'C'A'$.

8. $\therefore \text{area } \triangle ABC = \text{area } \triangle A'B'C'$.

686. Cor. *Two symmetrical triangles on the same or equal spheres are equal.*

1. Let $\triangle A''B''C''$ be symmetrical to $\triangle ABC$; let $\triangle A'B'C'$ be the spherical triangle on the same sphere as $\triangle ABC$, such that A' , B' , and C' are diametrically opposite to A , B , and C , respectively.

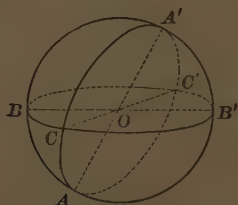
2. Then $\triangle A'B'C'$ is also symmetrical to $\triangle ABC$, and equal to $\triangle ABC$.

3. $\therefore$ the parts of $\triangle A''B''C''$ and $\triangle A'B'C'$ are equal and are arranged in the same order. Hence $\triangle A''B''C'' \cong \triangle A'B'C'$.

4. $\therefore \triangle ABC = \triangle A''B''C''$.

PROPOSITION XVI. THEOREM

687. *A spherical triangle of a sphere equals one half a lune of that sphere whose angle in degrees equals the spherical excess of the triangle.*



Hypothesis. A , B , and C are the measures in degrees of the angles of spherical $\triangle ABC$. E represents the spherical excess of the triangle.

Conclusion.

$$\triangle ABC = \frac{1}{2}L_E;$$

that is, $\triangle ABC =$ one half a lune whose angle is E .

Proof. 1. Complete the $\odot ABA'B'$, $BCB'C'$, and $ACA'C'$, and draw diameters AA' , BB' , and CC' .

$$2. \quad \triangle ABC + \triangle ACB' = \text{lune of } \angle B, \text{ or } L_B.$$

$$3. \quad \triangle ABC + \triangle A'CB = \text{lune of } \angle A, \text{ or } L_A.$$

$$4. \quad \triangle ABC + \triangle ABC' = \text{lune of } \angle C, \text{ or } L_C.$$

But $\triangle A'B'C = \triangle ABC'$, so

§ 685

$$5. \quad \triangle ABC + \triangle A'B'C = \text{lune of } \angle C, \text{ or } L_C.$$

$$6. \text{ Adding the equations of steps 2, 3, and 5,}$$

$$2\triangle ABC + (\triangle ABC + \triangle ACB' + \triangle A'CB + \triangle A'B'C)$$

$$= L_{(A+B+C)}.$$

§ 676

$$7. \text{ But } \triangle ABC + \triangle ACB' + \triangle A'CB + \triangle A'B'C$$

$$= \text{surface of hemisphere}$$

$$= L_{180}.$$

§ 678

$$8. \quad \therefore 2\triangle ABC + L_{180} = L_{A+B+C}.$$

$$9. \quad \therefore 2\triangle ABC = L_{A+B+C} - L_{180} = L_{(A+B+C)-180}.$$

$$10. \quad \text{But } A + B + C - 180 = E.$$

$$11. \quad \therefore 2\triangle ABC = L_E, \text{ or } \triangle ABC = \frac{1}{2}L_E.$$

688. Cor. 1. *If the radius of the sphere is R and the spherical excess of $\triangle ABC$ in degrees is E , then*

$$\text{area of } \triangle ABC = \frac{1}{2} L_E = \frac{1}{2} \frac{\pi R^2 E}{90} = \frac{\pi R^2 E}{180}.$$

689. Cor. 2. *The area of any spherical polygon whose excess is E is $\frac{\pi R^2 E}{180}$.*

Suggestion.—Divide the polygon into $\triangle$ by drawing diagonals from one vertex. Express the area of each triangle. Remember that the excess of the polygon equals the sum of the excesses of the triangles, when the polygon is divided as suggested.

Note.—A *spherical degree* may be defined as being a bi-rectangular spherical triangle whose third angle is one spherical angular degree. The area of a spherical triangle in spherical degrees can be proved to equal its spherical excess in degrees.

Ex. 59. Determine the area of a spherical triangle whose angles are 125° , 133° , and 156° , on a sphere whose radius is 10 in.

Ex. 60. What is the ratio of the areas of two spherical triangles on the same sphere whose angles are 94° , 135° , and 146° , and 87° , 105° , and 118° , respectively.

Ex. 61. Determine the area of a spherical triangle whose angles are 103° , 112° , and 127° on a sphere whose area is 160.

Ex. 62. Find the area of a spherical hexagon whose angles are 120° , 139° , 148° , 155° , 162° , and 167° , on a sphere whose radius is 12.

Ex. 63. The sides of a spherical triangle on a sphere whose radius is 15 in. are 44° , 63° , and 97° . Find the area of its polar triangle.

Ex. 64. Determine the part of the area of the surface of a sphere intercepted by a trihedral angle whose dihedral angles are 89° , 55° , and 100° .

Ex. 65. Express the ratio of a spherical triangle to the surface of the sphere in terms of the spherical excess E of the triangle, when (a) the excess is measured in degrees; (b) the excess is measured in right angles.

Ex. 66. The area of a spherical pentagon, four of whose angles are 112° , 131° , 138° , and 168° , is 27. If the area of the surface of the sphere is 120, what is the other angle?

Ex. 67. What part of the surface of a sphere is a tri-rectangular triangle of the sphere?

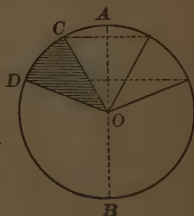
Ex. 68. Compare the area of a tri-rectangular spherical triangle of a sphere whose radius is 10 in. with the area of the plane triangle formed by the chords of the sides of the spherical triangle.

VOLUME OF A SPHERE

690. If a semicircle be revolved about its diameter as an axis, the solid generated by any sector of the semicircle is called a **Spherical Sector**.

Thus if semicircle $ACDB$ be revolved about diameter AB as an axis, sector OCD generates a spherical sector.

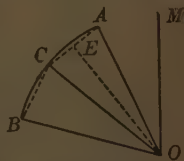
The zone generated by the arc of the circular sector is called the base of the spherical sector.



Note.—In the following pages, the expression “Vol. OCD ” will be used to denote the volume of the solid generated by revolving the portion of the plane within OCD around some axis specified.

691. Application of Limits to Spherical Sectors. Let O be the center of arc AB , and OM be any diameter of the circle whose center is O . Let C bisect arc AB , and draw radii OA , OB , and OC . Draw $OE \perp AC$; draw broken line ACB .

If the sector OAB is revolved about OM as an axis, it generates a spherical sector. The portion of the plane bounded by polygon $OACB$ generates a solid which is less than the spherical sector generated by circular sector OAB .

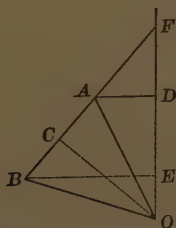


If arcs AC and CB are bisected at D and F respectively, and broken line $ADCFB$ is drawn, then when the figure is revolved about OM , the part of the plane bounded by polygon $OADCFB$ generates a solid more nearly equal to the spherical sector. If the process of halving the arcs be continued indefinitely, it will be assumed evident that the solid generated by the part of the plane bounded by the inscribed polygon approaches the spherical sector as limit.

Notice that the surface generated by broken line $ADCFB$ approaches as limit the zone generated by arc ACB . (§ 668.)

PROPOSITION XVII. THEOREM

692. *If a portion of a plane inclosed by an isosceles triangle be revolved about a straight line in its plane as axis, which passes through its vertex without intersecting its surface and without being parallel to its base, the volume of the solid generated is equal to the area of the surface generated by its base multiplied by one third its altitude.*



Hypothesis. Isosceles $\triangle OAB$, and the surface inclosed, are revolved about straight line OF in its plane; OF is not $\parallel$ base AB ; $OC \perp AB$.

Conclusion. Vol. $OAB = \text{area } AB \times \frac{1}{3} OC$.

Proof. 1. Draw $AD \perp OF$ and $BE \perp OF$; extend BA to meet OF at F .

$$2. \text{ Vol. } OBF = \text{vol. } OBE + \text{vol. } BEF$$

$$3. \quad = \frac{1}{3} \pi \overline{BE}^2 \times OE + \frac{1}{3} \pi \overline{BE}^2 \times EF \quad \S 616$$

$$4. \quad = \frac{1}{3} \pi \overline{BE}^2 (OE + EF) = \frac{1}{3} \pi BE \times BE \times OF.$$

$$5. \text{ But } BE \times OF = OC \times BF. \quad \text{Prove it.}$$

$$6. \quad \therefore \text{ vol. } OBF = \frac{1}{3} \pi BE \times OC \times BF.$$

$$7. \text{ But } \pi BE \times BF \text{ is the area } BF. \quad \S 614$$

$$8. \quad \therefore \text{ vol. } OBF = \text{area } BF \times \frac{1}{3} OC.$$

$$9. \text{ Similarly } \text{vol. } OAF = \text{area } AF \times \frac{1}{3} OC.$$

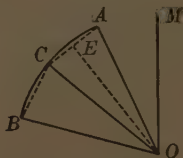
10. Subtracting 9 from 8,

$$\text{vol. } OAB = (\text{area } BF - \text{area } AF) \times \frac{1}{3} OC.$$

$$11. \quad = \text{area } AB \times \frac{1}{3} OC.$$

PROPOSITION XVIII. THEOREM

693. *The volume of a spherical sector is equal to the area of the zone which forms its base, multiplied by one third the radius of the sphere.*



Hypothesis. Sector OAB of $\odot O$ is revolved about diameter OM as an axis; R is the radius of the sphere.

Conclusion. Vol. of the spherical sector generated by circular sector OAB = area of zone generated by $\widehat{AB} \times \frac{1}{3} R$.

Proof. 1. Let C bisect $\widehat{AB}$. Draw AC , CB , OC ; and draw $OE \perp AC$.

$$2. \quad \text{Vol. } OAC = \text{area } AC \times \frac{1}{3} OE. \quad \S 692$$

$$3. \quad \text{Vol. } OCB = \text{area } CB \times \frac{1}{3} OE.$$

$$4. \quad \text{Adding, vol. } OACB = (\text{area } AC + \text{area } CB) \times \frac{1}{3} OE \\ = \text{area } ACB \times \frac{1}{3} OE.$$

5. Let the subdivisions of $\widehat{AB}$ be bisected indefinitely.

6. Then vol. $OACB \doteq$ vol. generated by sector OAB ,

and area $ACB \doteq$ area of zone generated by $\widehat{AB}$

and $OE \doteq R$.

$$7. \quad \therefore \text{vol. generated by sector } OAB \\ = \text{area of zone generated by } \widehat{AB} \times \frac{1}{3} R. \quad \S 403$$

694. Cor. 1. If v denotes the volume of a spherical sector, h the altitude of the zone which forms its base, and R the radius of the sphere,

$$v = 2 \pi R h \times \frac{1}{3} R = \frac{2}{3} \pi R^2 h. \quad \S 671$$

695. Cor. 2. The sphere may be regarded as a spherical sector whose base is the entire surface of the sphere. Letting

V denote the volume of the sphere, and R its radius,

$$V = 4\pi R^2 \times \frac{1}{3}R = \frac{4}{3}\pi R^3.$$

The volume of a sphere is equal to the cube of its radius multiplied by $\frac{4}{3}\pi$.

696. Cor. 3. *The volumes of two spheres have the same ratio as the cubes of their radii.*

Ex. 69. Find the volume of the sphere whose radius is 12.

Ex. 70. Determine the volume of metal in a spherical shell 10 in. in diameter and $\frac{1}{2}$ in. thick.

Ex. 71. A spherical cannon ball 9 in. in diameter is lowered into a cubical box filled with water, whose depth is 9 in. How many cubic inches of water will be left in the box?

Ex. 72. If a sphere 6 in. in diameter weighs 351 oz., what is the weight of a sphere of the same material whose diameter is 10 in.?

Ex. 73. The outer diameter of a spherical shell is 9 in., and its thickness is 1 in. What is the weight, if a cubic inch of the metal weighs one third pound?

Ex. 74. Find the area of the surface and the volume of the sphere inscribed in a cube the area of whose surface is 486 sq. in.

Ex. 75. Find the radius and the volume of a sphere, the area of whose surface is 324π sq. in.

Ex. 76. Prove that the volume of a sphere is two thirds the volume of its circumscribed cylinder.

Ex. 77. Within a sphere of radius R is inscribed a right circular cylinder whose altitude equals the diameter of its base.

(a) Determine its lateral area and compare the result with the area of the surface of the sphere.

(b) Compare its volume with the volume of the sphere.

Ex. 78. Given a spherical surface of radius R and its circumscribed right circular cylinder. From the center of the sphere, draw lines to the points of the circles bounding the bases of the cylinder, thus forming the two right circular cones. Compare the volume of the sphere with the difference between the volume of the cylinder and the sum of the volumes of the two cones.

Ex. 79. A cylindrical vessel, 8 in. in diameter, is filled to the brim with water. A ball is immersed in it, displacing water to the depth of $2\frac{1}{2}$ in. Find the diameter of the ball.

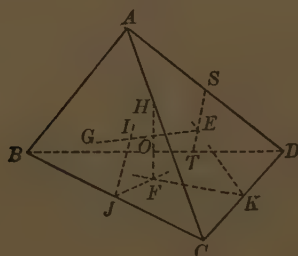
Note. — Supplementary Exercises 72–85, p. 460, can be studied now.

SUPPLEMENTARY TOPICS

GROUP A. CONSTRUCTION OF SPHERES

PROPOSITION XIX. THEOREM

697. *Through four points not in the same plane, one and only one spherical surface can be passed.*



Hypothesis. $A, B, C,$ and D are four points, not in the same plane.

Conclusion. One and only one spherical surface can be passed through $A, B, C,$ and D .

Proof. 1. Every point equidistant from C and D must lie in a plane $EKF \perp CD$ at its mid-point K ; and conversely.

§§ 457; 459

2. Every point equidistant from B and C must lie in a plane $IJF \perp BC$ at its mid-point J ; and conversely.

3. These planes intersect in a line HF , which is $\perp$ plane BCD at F , the circumcenter of $\triangle BCD$. § 499

Also, by steps 1 and 2, every point equidistant from $B, C,$ and D must lie in HF ; and conversely.

4. Similarly, every point equidistant from $A, C,$ and D lies in line GE , which is $\perp$ plane ACD at E , the circumcenter of $\triangle ACD$.

5. Since E and F are in plane EKF , then GE and HF lie in plane EKF . § 497

6. $\therefore GE$ intersects HF at a point O .

7. $\therefore O$ is one and the only point which is equidistant from A, B, C , and D . Steps 3 and 4

8. $\therefore$ a sphere with center O and radius OA is the one and only sphere through A, B, C , and D .

698. Cor. *A sphere may be circumscribed about any tetraedron.*

Ex. 80. What is the locus of the center of a sphere which will have a given radius r and will pass through a given point P ?

Ex. 81. What is the locus of the center of a sphere which will pass through each of two given points?

Ex. 82. What is the locus of the center of a sphere which will pass through each of three given points?

Ex. 83. What is the locus of the center of a sphere which will pass through all the points of a given circle?

Ex. 84. Is it possible for a sphere to pass through all the points of a given circle and also through a given point outside the plane of the circle? If so, tell how to determine its center and its radius.

Ex. 85. Prove that a sphere can be circumscribed about a cube.

Ex. 86. What is the locus of the center of a sphere which is tangent to a given plane at a given point?

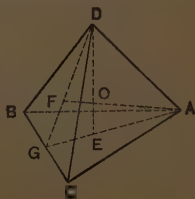
Ex. 87. What is the locus of the center of a sphere which will have a given radius r and will be tangent to a given plane?

Ex. 88. What is the locus of the center of a sphere which will be tangent to each of the faces of a given diedral angle?

Ex. 89. Is it possible for a sphere to be tangent to each of the faces of a given triedral angle? If there is more than one such sphere, what is the locus of the center?

Ex. 90. Find the area of the spherical surface passing through the vertices of a regular tetraedron whose edge is 8.

Suggestion.—Draw DOE and AOF perpendicular to $\triangle ABC$ and BCD respectively.



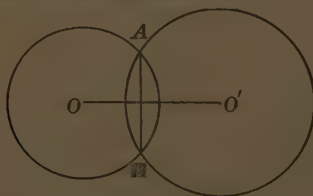
Ex. 91. Prove that a sphere can be inscribed in a given tetraedron.

GROUP B. GENERAL THEOREMS OF SPHERICAL GEOMETRY

699. Special interest attaches to the following theorems because of their similarity to certain theorems of plane geometry. In each case, the pupil should recall the corresponding theorem of plane geometry, if there is one, or should note the difference between the theorem of spherical geometry and the corresponding situation in plane geometry.

PROPOSITION XX. THEOREM

700. *The intersection of two spherical surfaces is a circle, whose center is in a straight line joining the centers of the spheres and whose plane is perpendicular to that line.*



Hypothesis. O and O' are the centers of two intersecting spherical surfaces.

Conclusion. The intersection of the surfaces is a circle whose center is in line OO' and whose plane is $\perp OO'$.

Proof. 1. Through O and O' and any point A of the intersection, pass a plane. This plane cuts the two surfaces in two intersecting great $\odot$. Let AB be the common chord of these two $\odot$, intersecting OO' at C .

2. $\therefore OO'$ bisects AB at right angles. § 207

3. If the entire figure is revolved about OO' as an axis, the $\odot$ will generate the spherical surfaces whose centers are O and O' .

Point A will generate a $\odot$ whose center is C and radius AC , which is common to the two spherical surfaces.

4 The plane of $\odot AC$ is $\perp OO'$. § 458

5. No point outside $\odot ACB$ can lie in both surfaces; for, if there were such a point, the two surfaces would necessarily coincide. § 697

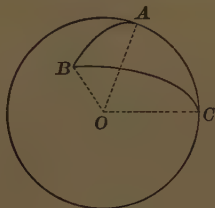
Ex. 92. What is the locus of points at the distance r_1 from a given point P_1 and at the distance r_2 from a given point P_2 ?

Ex. 93. The distance between the centers of two spheres whose radii are 25 and 17, respectively, is 28. Find the diameter of their circle of intersection, and its distance from the center of each sphere.

Suggestion. — Recall § 313.

PROPOSITION XXI. THEOREM

701. *Any side of a spherical triangle is less than the sum of the other two.*



Hypothesis. AB is any side of spherical $\triangle ABC$.

Conclusion. $\widehat{AB} < \widehat{AC} + \widehat{BC}$.

Suggestions. — 1. Compare $\angle AOB$ with $\angle AOC + \angle BOC$.

2. What is the relation of the measure of $\widehat{AB}$ and that of $\angle AOB$? Of $\widehat{AC}$ and $\angle AOC$? etc.

Ex. 94. Prove that any side of a convex spherical polygon is less than the sum of the remaining sides.

Ex. 95. Prove that the sum of the arcs of great circles drawn from any point within a spherical triangle to the extremities of any side, is less than the sum of the other two sides of the triangle.

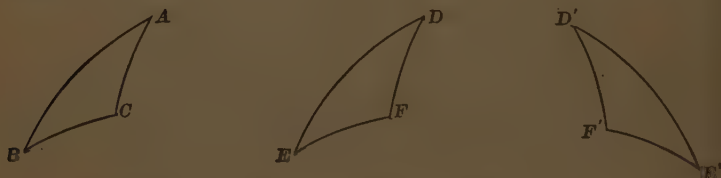
702. Two spherical polygons on the same sphere or equal spheres are mutually equilateral or mutually equiangular when the sides or angles of one are equal respectively to the sides or angles of the other, whether taken in the same or in opposite orders.

PROPOSITION XXII. THEOREM

703. *If two spherical triangles on the same sphere, or equal spheres, have two sides and the included angle of one equal respectively to two sides and the included angle of the other,*

I. *They are congruent if the equal parts occur in the same order.*

II. *They are symmetrical if the equal parts occur in opposite orders.*



I. **Hypothesis.** ABC and DEF are spherical Δ on the same sphere, or equal spheres, having $\widehat{AB} = \widehat{DE}$, $\widehat{AC} = \widehat{DF}$, and $\angle A = \angle D$; and the equal parts occur in the same order.

Conclusion. $\Delta ABC \cong \Delta DEF$.

Suggestion. — Prove it by superposition as in § 63.

II. **Hypothesis.** ABC and $D'E'F'$ are spherical Δ on the same sphere, or equal spheres, having $\widehat{AB} = \widehat{D'E'}$, $\widehat{AC} = \widehat{D'F'}$, and $\angle A = \angle D'$; and the equal parts occur in opposite orders.

Conclusion. ABC and $D'E'F'$ are symmetrical Δ .

Proof. 1. Let DEF be a spherical Δ on the same sphere, or an equal sphere, symmetrical to $\Delta D'E'F'$, having

$$\widehat{DE} = \widehat{D'E'}, \quad \widehat{DF} = \widehat{D'F'}, \quad \text{and} \quad \angle D = \angle D',$$

the equal parts occurring in opposite orders.

2. Then in spherical ΔABC and DEF ,

$$\widehat{AB} = \widehat{DE}, \quad \widehat{AC} = \widehat{DF}, \quad \text{and} \quad \angle A = \angle D;$$

and the equal parts occur in the same order.

3. $\therefore \Delta ABC \cong \Delta DEF$.

4. $\therefore \Delta ABC$ is symmetrical to $\Delta D'E'F'$.

Ex. 96. Prove that the arc of a great circle bisecting the vertical angle of an isosceles spherical triangle is perpendicular to the base and bisects the base.

Ex. 97. Prove that the angles opposite the equal sides of an isosceles spherical triangle are equal.

PROPOSITION XXIII. THEOREM

704. *If two spherical triangles on the same sphere, or on equal spheres, have a side and two adjacent angles of one equal respectively to a side and two adjacent angles of the other,*

- I. *They are congruent if the equal parts occur in the same order.*
- II. *They are symmetrical if the equal parts occur in opposite orders.*

The proof is left to the student.

PROPOSITION XXIV. THEOREM

705. *If two spherical triangles on the same sphere, or on equal spheres, are mutually equilateral, they are mutually equiangular.*



Hypothesis. $\triangle ABC$ and $\triangle DEF$ are mutually equilateral spherical $\triangle$ on the equal sphere; $\widehat{BC}$ and $\widehat{EF}$ are homologous.

Conclusion. $\triangle ABC$ and $\triangle DEF$ are mutually equiangular.

Suggestions. — 1. Let O and O' be the centers of the respective spheres, and draw the radii to A, B, C, D, E , and F . Consider the two trihedral angles $O-ABC$ and $O'-DEF$.

2. Compare the face angles of trihedrals $O-ABC$ and $O'-DEF$.

3. Compare the dihedral angles of $O-ABC$ and $O'-DEF$.

4. Now compare the $\angle A$ and D ; also the $\angle B$ and E ; also $\angle C$ and F .

Note. — The theorem may be proved in a similar manner when the given spherical $\triangle$ are on the same sphere.

706. Cor. *If two spherical triangles on the same sphere, or on equal spheres, are mutually equilateral,*

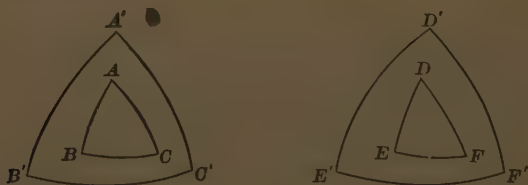
I. *They are congruent if the equal parts occur in the same order.*

II. *They are symmetrical if the equal parts occur in opposite orders.*

Ex. 98. Prove that the arc of a great circle drawn from the vertex of an isosceles spherical triangle to the middle point of the base, is perpendicular to the base, and bisects the vertical angle.

PROPOSITION XXV. THEOREM

707. *If two spherical triangles on the same sphere, or equal spheres, are mutually equiangular, their polar triangles are mutually equilateral.*



Hypothesis. ABC and DEF are mutually equiangular spherical Δ on the same sphere or equal spheres, $\angle A$ and D being homologous; also, $\Delta A'B'C'$ is the polar Δ of ΔABC , and $\Delta D'E'F'$ of ΔDEF , A being the pole of $\widehat{B'C'}$, and D of $\widehat{E'F'}$.

Conclusion. $\Delta A'B'C'$ and $\Delta D'E'F'$ are mutually equilateral.

Suggestions. — 1. Compare the measure of $\angle A$ and $\widehat{B'C'}$; of $\angle D$ and $\widehat{E'F'}$. Then compare $\widehat{B'C'}$ and $\widehat{E'F'}$.

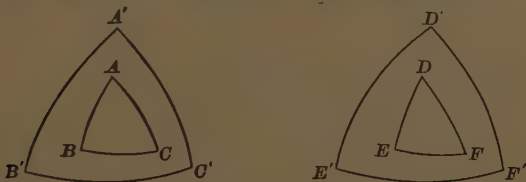
2. Proceed similarly for the other pairs of homologous sides.

708. Cor. *If two spherical triangles on the same sphere, or equal spheres, are mutually equilateral, their polar triangles are mutually equiangular.*

Suggestion. — Model the proof after that of § 707.

PROPOSITION XXVI. THEOREM

709. *If two spherical triangles on the same sphere, or on equal spheres, are mutually equiangular, they are mutually equilateral.*



Hypothesis. $\triangle ABC$ and $\triangle DEF$ are mutually equiangular spherical $\triangle$ on the same sphere or equal spheres.

Conclusion. $\triangle ABC$ and $\triangle DEF$ are mutually equilateral.

Proof. 1. Let $\triangle A'B'C'$ be the polar $\triangle$ of $\triangle ABC$, and $\triangle D'E'F'$ of $\triangle DEF$.

2. Since $\triangle ABC$ and $\triangle DEF$ are mutually equiangular, $\triangle A'B'C'$ and $\triangle D'E'F'$ are mutually equilateral. § 707

3. $\therefore \triangle A'B'C'$ and $\triangle D'E'F'$ are mutually equiangular.

4. $\therefore$ But $\triangle ABC$ is the polar $\triangle$ of $\triangle A'B'C'$ and $\triangle DEF$ of $\triangle D'E'F'$. § 655

5. $\therefore \triangle ABC$ and $\triangle DEF$ are mutually equilateral. Why?

710. Cor. *If two spherical triangles on the same sphere or on equal spheres are mutually equiangular,*

I. *They are congruent if the equal parts are arranged in the same order.*

II. *They are symmetrical if the equal parts are arranged in opposite orders.*

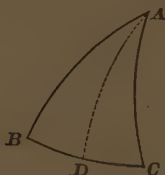
Ex. 99. Compare the theorem of Proposition XXVI with the corresponding theorem about two plane triangles.

Ex. 100. If three diameters of a sphere be drawn so that each is perpendicular to the other two, the planes determined by them divide the surface of the sphere into eight congruent tri-rectangular triangles.

Note. — Recall at this point all the theorems by which two spherical triangles can be proved congruent.

PROPOSITION XXVII. THEOREM

711. *In an isosceles triangle, the angles opposite the equal sides are equal.*



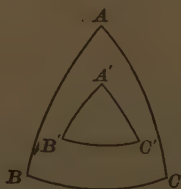
Hypothesis. In spherical triangle ABC , $\widehat{AB} = \widehat{AC}$.

Conclusion. $\angle B = \angle C$.

Suggestion. — Let $\widehat{AD}$, an arc of a great circle, bisect $\widehat{BC}$. Use § 705.

PROPOSITION XXVIII. THEOREM

712. *If two angles of a spherical triangle are equal, the sides opposite are equal.*



Hypothesis. In spherical $\triangle ABC$, $\angle B = \angle C$.

Conclusion. $\widehat{AB} = \widehat{AC}$.

Suggestions. — 1. Let $\triangle A'B'C'$ be the polar $\triangle$ of ABC , B being the pole of $\widehat{A'C'}$, and C of $\widehat{A'B'}$.

2. Compare $\widehat{A'C'}$ and $\widehat{A'B'}$ by using § 656.

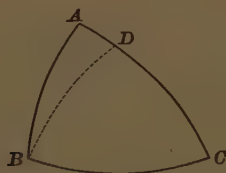
3. Compare $\angle B'$ and $\angle C'$, and from them determine the relation between $\widehat{AB}$ and $\widehat{AC}$.

Ex. 101. Prove that the great circle arcs drawn to the extremities of an arc of a great circle from any point on the great circle perpendicular to and bisecting it are equal.

Ex. 102. State and prove the converse of Ex. 101.

PROPOSITION XXIX. THEOREM

713. *If two angles of a spherical triangle are unequal, the sides opposite them are unequal, the side opposite the greater angle being the greater.*



Hypothesis. In spherical $\triangle ABC$, $\angle ABC > \angle C$.

Conclusion. $\widehat{AC} > \widehat{AB}$.

Proof. 1. Let $\widehat{BD}$, an arc of a great circle, meeting $\widehat{AC}$ at D , make $\angle CBD = \angle C$.

2. $\therefore \widehat{BD} = \widehat{DC}$. Why?

3. $\widehat{AD} + \widehat{BD} > \widehat{AB}$. Why?

4. $\therefore \widehat{AD} + \widehat{DC} > \widehat{AB}$.

5. $\therefore \widehat{AC} > \widehat{AB}$.

714. Cor. *If two sides of a spherical triangle are unequal, the angles opposite are unequal, the angle opposite the greater side being the greater.*

An indirect proof based upon §§ 712 and 713 is quite easy.

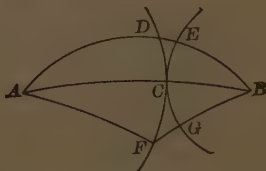
Ex. 103. What is the locus of points on the surface of a sphere which are equidistant from the extremities of an arc of a great circle of that sphere?

Ex. 104. Prove that the great circle arcs perpendicular to and bisecting the sides of a spherical triangle intersect in a point which is equidistant from the vertices of the triangle.

Ex. 105. Prove that a circle can be circumscribed about a spherical triangle.

PROPOSITION XXX. THEOREM

715. *The shortest line on the surface of a sphere between two given points is the arc of a great circle, not greater than a semicircle which joins the two points.*



Hypothesis. Points A and B are on the surface of a sphere, and $\widehat{AB}$ is an arc of a great $\odot$, not greater than a semicircle.

Conclusion. $\widehat{AB}$ is the shortest line on the surface of the sphere between A and B .

Note.—The following proof is divided into four parts, (a), (b), (c), and (d).

Proof. (a) 1. Let C be any point in $\widehat{AB}$.

2. Let $\widehat{DCF}$ and $\widehat{ECG}$ be arcs of small $\odot$ with A and B respectively as poles, and $\widehat{AC}$ and $\widehat{BC}$ as polar distances.

(b) $\widehat{DCF}$ and $\widehat{ECG}$ have only point C in common.

1. For let F be any other point in $\widehat{DCF}$ and draw $\widehat{AF}$ and $\widehat{BF}$, arcs of great circles.

2. $\therefore \widehat{AF} = \widehat{AC}$. § 643

3. But $\widehat{AF} + \widehat{BF} > \widehat{AC} + \widehat{BC}$. Why?

4. Subtracting $\widehat{AF}$ from the first member of the inequality and its equal $\widehat{AC}$ from the second member,

$$\widehat{BF} > \widehat{BC}, \text{ or } \widehat{BF} > \widehat{BG}.$$

5. $\therefore F$ lies outside small $\odot ECG$, and $\widehat{DCF}$ and $\widehat{ECG}$ have only point C in common.

(c) The shortest line on the surface of the sphere from A to B must pass through C .

1. Let $ADEB$ be any line on the surface of the sphere between A and B , not passing through C , and cutting $\widehat{DCF}$ and $\widehat{ECG}$ at D and E respectively.

2. Then, whatever the nature of line AD , it is evident that an equal line can be drawn from A to C .

3. In like manner, whatever the nature of line BE , an equal line can be drawn from B to C .

4. Hence a line can be drawn from A to B passing through C , equal to the sum of lines AD and BE , and consequently less than $ADEB$ by the part DE .

5. Therefore no line which does not pass through C can be the shortest line between A and B .

(d) $\widehat{AB}$ is the shortest line from A to B on the surface of the sphere.

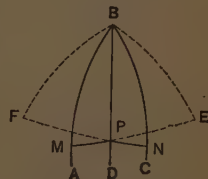
1. But C is any point in $\widehat{AB}$.

2. Hence the shortest line from A to B must pass through every point of $\widehat{AB}$.

3. Then the great circle arc AB is the shortest line on the surface of the sphere between A and B .

Ex. 106. Prove that any point in the arc of a great circle bisecting a spherical angle is equally distant (§ 573) from the sides of the angle.

(Prove $\widehat{PM} = \widehat{PN}$. Let E be a pole of arc AB , and F of arc BC . Spherical $\triangle BPE$ and BPF are symmetrical by § 702, II., and $\widehat{PE} = \widehat{PF}$.)



Ex. 107. Prove that a point on the surface of a sphere, equally distant from the sides of a spherical angle, lies in the arc of a great circle bisecting the angle.

(Fig. of Ex. 118. Prove $\angle ABP = \angle CBP$. Spherical $\triangle BPE$ and BPF are symmetrical by § 706.)

Ex. 108. What is the locus of points on the surface of a sphere equally distant from the sides of a spherical angle?

Ex. 109. Prove that the arcs of great circles bisecting the angles of a spherical triangle meet in a point equally distant from the sides of the triangle.

Ex. 110. Prove that a circle may be inscribed in any spherical triangle.

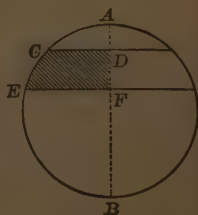
GROUP C. SPHERICAL SEGMENTS, PYRAMIDS, AND WEDGES

716. A **Spherical Segment** is the portion of a sphere included between two parallel planes which intersect the sphere.

The portions of the planes bounding the segment are the *bases* of the segment; the perpendicular between the planes is the *altitude* of the segment.

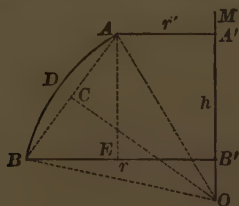
A spherical segment of one base is the spherical segment one of whose bounding planes is a tangent to the sphere.

If a semicircle $ACEB$ be revolved about diameter AB as an axis, and CD and EF are perpendicular to AB , the portion of the plane bounded by $FECD$ generates a spherical segment whose altitude is DF , and whose bases have radii CD and EF respectively; the portion ACD generates a spherical segment of one base whose altitude is AD .



717. If r and r' are the radii of the bases, h the altitude, and v the volume of a spherical segment, then

$$v = \frac{1}{2} \pi (r^2 + r'^2) h + \frac{1}{8} \pi h^3.$$



Let O be the center of $\widehat{ADB}$; let AA' and BB' be $\perp$ to the diameter OM ; let $AA' = r'$, $BB' = r$, $A'B' = h$. Let the whole figure revolve about OM as an axis. Let v = the volume of the resulting spherical segment.

Solution. 1. Draw OA , OB , and AB ; draw $OC \perp AB$, and $AE \perp BB'$. Let $OA = R$

2. Now, vol. $ADBB'A' = \text{vol. } ACBD + \text{vol. } ABB'A'. \quad (1)$
3. Also, $\text{vol. } ACBD = \text{vol. } OADB - \text{vol. } OAB.$
4. $\text{vol. } OADB = \frac{2}{3} \pi R^2 h. \quad \S 694$
5. And, $\text{vol. } OAB = \text{area } AB \times \frac{1}{3} OC \quad \S 691$
6. $= h \times 2 \pi OC \times \frac{1}{3} OC \quad \S 668$
7. $= \frac{2}{3} \pi \overline{OC}^2 h.$
8. $\therefore \text{vol. } ACDB = \frac{2}{3} \pi R^2 h - \frac{2}{3} \pi \overline{OC}^2 h$
9. $= \frac{2}{3} \pi (R^2 - \overline{OC}^2) h.$
10. But, $R^2 - \overline{OC}^2 = \overline{AC}^2 \quad \text{Why?}$
11. $= (\frac{1}{2} AB)^2$
12. $= \frac{1}{4} \overline{AB}^2.$
13. $\therefore \text{vol. } ACDB = \frac{2}{3} \pi \times \frac{1}{4} \overline{AB}^2 \times h = \frac{1}{6} \pi \overline{AB}^2 h.$
14. Now, $\overline{AB}^2 = \overline{BE}^2 + \overline{AE}^2$
15. $= (r - r')^2 + h^2.$
16. $\therefore \text{vol. } ACDB = \frac{1}{6} \pi [(r - r')^2 + h^2] h.$
17. Also, $\text{vol. } ABB'A' = \frac{1}{3} \pi (r^2 + r'^2 + rr') h. \quad \S 621$
18. Substituting in step 2, vol. $ADBB'A'$
 $= \frac{1}{6} \pi [(r - r')^2 + h^2] h + \frac{1}{6} \pi (2r^2 + 2r'^2 + 2rr') h$
19. $= \frac{1}{6} \pi (r^2 - 2rr' + r'^2 + h^2 + 2r^2 + 2r'^2 + 2rr') h$
20. $= \frac{1}{6} \pi (3r^2 + 3r'^2) h + \frac{1}{6} \pi h^3$
21. $= \frac{1}{2} \pi (r^2 + r'^2) h + \frac{1}{6} \pi h^3.$

Ex. 111. Find the volume of a spherical segment, the radii of whose bases are 4 and 5, and whose altitude is 9.

718. A **Spherical Wedge** is a solid bounded by a lune and the planes of its bounding arcs.

Evidently, two spherical wedges in the same sphere, or equal spheres, are congruent when their angles are equal.

Also, it is evident that two wedges in the same sphere can be added by placing them so that they have one common bounding plane. The angle of the sum is equal to the sum of the angles of the wedges.

Note. — Review at this time § 675 and § 676, noting the analogy between wedges and lunes.

719. It can be proved as in § 677 that two wedges have the same ratio as their angles. (Cf. § 677.)

720. A sphere may be regarded as a wedge whose angle is 360° . (Cf. § 678.)

Therefore, a wedge of a sphere whose radius is r , whose angle contains A degrees has a volume v determined as follows:

$$\frac{v}{\frac{4}{3}\pi r^3} = \frac{A}{360}, \text{ or } v = \frac{\pi r^3 A}{270}.$$

721. Since the lune whose angle is A degrees, on a sphere whose radius is r , has its area expressed by the formula $\frac{\pi r^2 A}{90}$, (§ 680)

$\therefore$ the volume of a wedge equals one third the radius of the sphere multiplied by the area of the lune which forms its base.

722. A **Spherical Pyramid** is a solid bounded by the spherical polygon and the planes of its sides; as $O-ABCD$ in the adjoining figure.

The center of the sphere is the *vertex* of the pyramid, and the spherical polygon is its *base*.

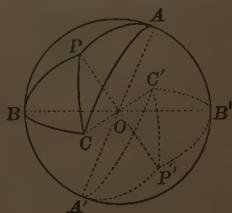
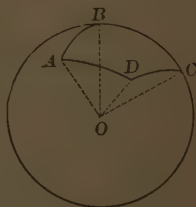
Two spherical pyramids are congruent when their bases are congruent, for they can be made to coincide.

723. Two spherical pyramids whose bases are symmetrical isosceles spherical triangles are congruent, for their bases are congruent by § 683.

724. Two spherical pyramids corresponding to a pair of vertical triedral angles are equal. (Cf. § 685.)

Suggestions. — 1. Recall the proof of § 685.

2. Compare spherical pyramids $O-APB$, $O-BPC$, and $O-CPA$ with spherical pyramids $O-A'P'B'$, $O-B'P'C'$, and $O-C'P'A'$, respectively.



725. *The volume of a triangular spherical pyramid equals one half the volume of a spherical wedge whose angle is the spherical excess of the base of the pyramid.*

The proof is exactly like that for § 687.

726. *If the radius of the sphere is r and the excess of the base of a triangular spherical pyramid is E , and the volume of the spherical pyramid is v , then*

$$v = \frac{1}{2} \frac{\pi r^3 \cdot E}{270} = \frac{\pi r^3 E}{540}. \quad \S\ 720$$

727. The same formula may be employed to find the volume of any spherical pyramid, with the understanding that E is the spherical excess of the base of the pyramid, measured in degrees.

728. In the case of any spherical pyramid, the area of the base is $\frac{\pi r^2 E}{180}$ (§ 689). Hence the volume of any spherical pyramid is one third the area of its base multiplied by the radius of the sphere.

Ex. 124. Find the volume of a triangular spherical pyramid the angles of whose base are 92° , 119° , and 134° , if the volume of the sphere is 192.

Ex. 125. Find the volume of a quadrangular spherical pyramid, the angles of whose base are 107° , 118° , 134° , and 146° , if the diameter of the sphere is 12.

Ex. 126. The volume of a triangular spherical pyramid, the angles of whose base are 105° , 126° , and 147° , is $60\frac{1}{2}$. What is the volume of the sphere?

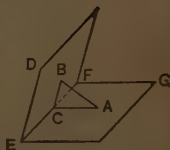
Ex. 127. Find the volume of a pentagonal spherical pyramid the angles of whose base are 109° , 128° , 137° , 153° , and 158° , if the volume of the sphere is 180.

Ex. 128. The volume of a quadrangular spherical pyramid, the angles of whose base are 110° , 122° , 135° , and 146° is $12\frac{3}{4}$. What is the volume of the sphere?

Ex. 129. What is the angle of the base of a spherical wedge whose volume is $\frac{4}{3}\pi$, if the radius of the sphere is 4?

SUPPLEMENTARY EXERCISES

Ex. 1. Two planes DEF and GEF intersect in line EF . A is any point in plane GEF . If AC be drawn perpendicular to EF , and AB perpendicular to plane DEF , prove the plane determined by AC and BC perpendicular to EF .



Ex. 2. Prove that the line joining the mid-points of one pair of opposite sides of a quadrilateral in space bisects the line joining the mid-points of the other pair of sides.

Ex. 3. If two intersecting planes pass through two parallel lines, their intersection is parallel to the parallel lines.

Ex. 4. If three planes intersect in pairs, the lines of intersection are either parallel or concurrent.

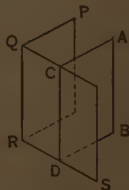
Suggestions.—Case (a) 1. Assume that two lines of intersection meet at a point P . 2. Prove that P is on the third line of intersection. Case (b) 1. Assume that two lines of intersection are parallel. 2. Prove that the third line of intersection is parallel to the other two by an indirect proof.

Ex. 5. Prove that a line parallel to each of two intersecting planes is parallel to their intersection.

Hyp. AB is $\parallel$ planes PR and QS .

Con. $AB \parallel QR$.

Suggestion.—Pass a plane through $AB \parallel PR$; then use § 471.

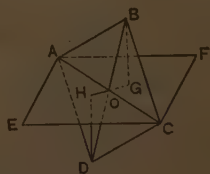


Ex. 6. If a plane be drawn through a diagonal of a parallelogram the perpendiculars to it from the extremities of the other diagonal are equal.

Hyp. $ABCD$ is a $\square$.

BG and $DH \perp$ plane $AECF$.

Con. $BG = DH$.



Ex. 7. D is any point in perpendicular AF from A to side BC of triangle ABC . If line DE be drawn perpendicular to the plane of ABC , and line GH be drawn through E parallel to BC , prove line AE perpendicular to GH .



Suggestion.—Prove $BC \perp$ to plane AED and then $GH \perp$ plane AED .

Ex. 8. (a) Through a line which is parallel to a plane, a plane can be drawn parallel to the given plane.

(b) Is the construction possible if the given line is not parallel to the given plane?

Ex. 9. If a plane, parallel to the edge of a diedral angle, intersects the faces of the angle, its intersections with the faces are parallel to the edge and to each other.

Ex. 10. If a straight line and a plane are both perpendicular to a given plane, they are parallel, unless the line lies in the plane.

Hyp. $CB \perp \text{plane } PR$; $\text{plane } MN \perp \text{plane } PR$.

Con. $CB \parallel MN$.

Suggestions. — 1. Draw $CD \perp RQ$, and let the plane determined by CB and CD meet MN in DE .

2. Prove $CB \parallel DE$ and hence $\parallel \text{plane } MN$ by § 466.

Ex. 11. If a plane is perpendicular to one of two perpendicular planes, its intersection with the other plane is also perpendicular to the first plane.

Ex. 12. From any point E within diedral angle $CABD$, EF and EG are drawn perpendicular to faces ABC and ABD , respectively, and GH perpendicular to face ABC at H . Prove FH is perpendicular to AB .

Suggestion — Prove that FH lies in the plane of EF and EG , by § 498; also consider the relation of AB and plane GEF .

Ex. 13. If BC is the projection of line AB upon plane MN , and BD and BE be drawn in the plane making $\angle CBD = \angle CBE$, prove $\angle ABD = \angle ABE$.

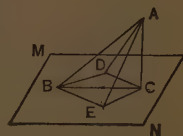
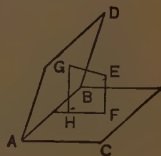
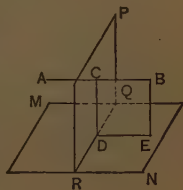
Suggestions. — 1. Lay off $BD = BE$, and draw lines AD , AE , CD , and CE .

2. Prove $\triangle ABD$ and $\triangle ABE$ congruent.

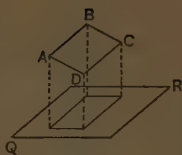
Ex. 14. If a line is perpendicular to one of two intersecting planes, its projection upon the other is perpendicular to the intersection of the two planes. (See the figure for Ex. 12.)

Suggestion. — If $EG \perp \text{plane } AD$, and FH is the projection of EG on plane BC , prove $FH \perp AB$.

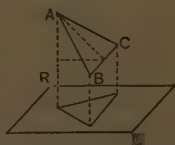
Ex. 15. If a straight line intersects two parallel planes, it makes equal angles with them.



Ex. 16. The base of a rectangle $ABCD$ is 10, and its altitude 8. Side 10 is parallel to plane QR . Side 8 makes an angle of 60° with QR . Find the area of the projection of $\square ABCD$ on plane QR , correct to three decimal places.



Ex. 17. An equilateral $\triangle ABC$, whose area is 25, has its side BC parallel to a plane QR . The plane of $\triangle ABC$ makes an angle of 45° with plane QR . Find the area of the projection of $\triangle ABC$ on plane RQ .



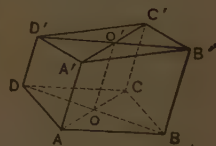
Ex. 18. Find the lateral area of a regular triangular prism each side of whose base is 5 and whose altitude is 8.

Ex. 19. Prove that the upper base of a truncated parallelopiped is a parallelogram.

Ex. 20. Prove that the sum of two opposite lateral edges of a truncated parallelopiped is equal to the sum of the other two lateral edges.

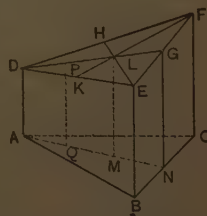
Suggestions. — 1. What kind of figure is $AA'C'C$?

2. Compare $AA' + CC'$ with OO' .



Ex. 21. Prove that the perpendicular drawn to the lower base of a truncated right triangular prism from the intersection of the medians of the upper base, is equal to one third the sum of the lateral edges.

Suggestion. — Let P be the mid-point of DL , and draw $PQ \perp ABC$; express LM in terms of PQ and GN .



Ex. 22. Prove that the sum of the squares of the four diagonals of a parallelopiped is equal to the sum of the squares of its twelve edges.

Suggestion. — Recall Ex. 142, Book III, p. 184.



Ex. 23. Determine the approximate area of the base of a bin 6 ft. deep that will hold 250 bu. of grain. (One bu. = 2150.42 cu. in.)

Ex. 24. Find the edge of a cube equivalent to a rectangular parallelopiped whose dimensions are 9 in., 1 ft. 9 in., and 4 ft. 1 in.

Ex. 25. Find the volume of a rectangular parallelopiped, the dimensions of whose base are 14 and 9, and the area of whose entire surface is 620.

Ex. 26. The diagonal of a cube is $8\sqrt{3}$. Find its volume, and the area of its entire surface.

Suggestion. — Represent the length of the edge by x .

Ex. 27. Find the dimensions of the base of a rectangular parallelopiped, the area of whose entire surface is 320, volume 336, and altitude 4.

Suggestion. — Represent the dimensions of the base by x and y .

Ex. 28. Find the area of the entire surface of a rectangular parallelopiped, the dimensions of whose base are 11 and 13, and volume 858.

Ex. 29. A trench is 124 ft. long, $2\frac{1}{2}$ ft. deep, 6 ft. wide at the top, and 5 ft. wide at the bottom. How many cubic feet of water will it contain?

Ex. 30. Prove that the volume of any oblique prism is equal to the product of the area of a right section by the length of a lateral edge.

Ex. 31. Prove that the volume of a regular prism is equal to its lateral area multiplied by one half the apothem of the base.

Ex. 32. The volume of a right prism is 2310, and its base is a right triangle whose legs are 20 and 21, respectively. Find its lateral area.

Ex. 33. The lateral area and volume of a regular hexagonal prism are 60 and $15\sqrt{3}$, respectively. Find its altitude, and one side of its base.

Suggestion. — Represent the altitude by x , and the side of the base by y .

Ex. 34. The altitude of a pyramid is 20 in., and its base is a rectangle whose dimensions are 10 in. and 15 in., respectively. What is the distance from the vertex of a section parallel to the base, whose area is 54 sq. in.?

Ex. 35. At what distance from the vertex must a plane parallel to the base be drawn so that the area of the section will be one half the base?

Ex. 36. In Ex. 35, replace the fraction $\frac{1}{2}$ by the fraction $\frac{1}{3}$ and solve the resulting exercise.

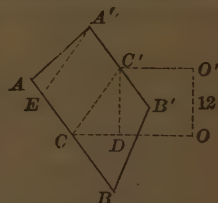
Ex. 37. In Ex. 35, replace the fraction $\frac{1}{2}$ by the fraction $\frac{2}{3}$ and solve the resulting exercise.

Ex. 38. Prove that the volume of a regular pyramid is equal to its lateral area, multiplied by one third the distance from the center of its base to any lateral face.

Suggestion. — Pass planes through the lateral edges and the center of the base.

Ex. 39. Find the lateral edge, lateral area, and volume of a frustum of a regular quadrangular pyramid, the sides of whose bases are 17 and 7, respectively, and whose altitude is 12.

Suggestion.—Let $ABB'A'$ be a lateral face of the frustum, and O and O' the centers of the bases; draw lines $OC \perp AB$, $O'C' \perp A'B'$, $C'D \perp OC$, and $A'E \perp AB$; also lines OO' and CC' .



Ex. 40. The bases of a frustum of a pyramid are rectangles, whose sides are 27 and 15, and 9 and 5, respectively, and the line joining their centers is perpendicular to each base. If the altitude of the frustum is 12, find its lateral area and volume.

Ex. 41. Find the lateral area and volume of a frustum of a regular triangular pyramid, the sides of whose bases are 12 and 6, respectively, and whose lateral edge is 5.

* **Ex. 42.** The altitude and lateral edge of a frustum of a regular triangular pyramid are 8 and 10, respectively, and each side of its upper base is $2\sqrt{3}$. Find its volume and lateral area.

Ex. 43. Find the volume of the rectangular prismoid the sides of whose bases are 10 and 7, and 6 and 5, respectively, and whose altitude is 9.

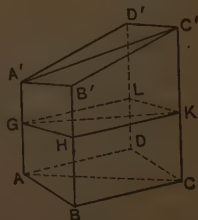
Ex. 44. The volume of a triangular prism is equal to a lateral face, multiplied by one half its perpendicular distance from any point in the opposite lateral edge.

Suggestion.—Draw a rt. section of the prism, and apply § 577.

Ex. 45. Prove that the volume of a truncated parallelopiped is equal to the area of a right section multiplied by one fourth the sum of the lateral edges.

Ex. 46. Prove that a plane passed through the center of a parallelopiped divides it into two equal solids.

Ex. 47. The volume of a truncated parallelopiped is equal to the area of a right section, multiplied by the distance between the centers of the bases.



Suggestion.—By Ex. 45, the distance between the centers of the bases may be proved equal to one fourth the sum of the lateral edges.

Ex. 48. How many square feet of heating surface are there in a hot-water conducting pipe 9 feet long and 2 inches in outside diameter?

Ex. 49. Determine the lateral area of the right circular cylinder formed by revolving a rectangle, having base b and altitude h ,
(a) about its base; (b) about its altitude.

Ex. 50. The lateral area of a cylinder of revolution is 120π . The area of the base is 36π . Find the altitude.

Ex. 51. The cross section of a tunnel, $2\frac{1}{2}$ mi. in length, is in the form of a rectangle 6 yd. wide and 4 yd. high, surmounted by a semicircle whose diameter is equal to the width of the rectangle; how many cubic yards of material were taken out in its construction? ($\pi = 3.1416$.)

Ex. 52. What must be the length in inches of a 10-gal. gasoline tank which is 10 in. in diameter?

Ex. 53. Determine the volume generated when a rectangle of base b and altitude h

(a) revolves about its side b ; (b) revolves about its side h .

Ex. 54. Two right circular cylinders have equal altitudes, but the radius of the base of the one is double the radius of the base of the other. Compare (a) their lateral areas; (b) their volumes.

Ex. 55. A regular hexagonal prism is inscribed in a right circular cylinder whose altitude is 10 in. and the radius of whose base is 3 in. Determine the difference between the volumes of the prism and cylinder.

Ex. 56. Prove that the volume of a cylinder of revolution is equal to its lateral area multiplied by one half the radius of its base.

Ex. 57. Express by a formula the volume of a round cast-iron column of length l ft., thickness t in., and outside diameter d in.

Ex. 58. Given the radius of the base R and the total area T of a cylinder of revolution, find its volume.

(Find H from the equation $T = 2\pi RH + 2\pi R^2$.)

Ex. 59. Given the diameter of the base D and the volume V of a cylinder of revolution, find its lateral area and total area.

Ex. 60. The volume of a circular cone is V . What is the effect upon the volume:

(a) if the radius of the base is doubled?

(b) if the altitude is doubled?

(c) if both the radius and the altitude are doubled?

Ex. 61. The altitude of a cone of revolution is 27 in., and the radius of its base is 16 in. What is the diameter of the base of an equal cylinder, whose altitude is 16 in.?

Ex. 62. A plane is passed parallel to the base of a circular cone so as to bisect the altitude. What is the ratio of the two parts into which the given cone is divided?

Ex. 63. Determine the lateral area of a right circular cone whose volume is 320π cu. in., and whose altitude is 15 in.

Ex. 64. Determine the volume of a cone of revolution whose slant height is 29 in., and whose lateral area is 580π sq. in.

Ex. 65. If the altitude of a cone of revolution is three fourths the radius of its base, the volume is equal to its lateral area multiplied by one fifth the radius of its base.

Ex. 66. Given the altitude H and the volume V of a right circular cone. Derive the formula for the lateral area in terms of V and H .

Ex. 67. Given the slant height L and the lateral area S of a right circular cone. Derive the formula for its volume in terms of S and L .

Ex. 68. Find the lateral area of the frustum of a right circular cone, whose altitude is 8 in., if the radii of its bases are 6 in. and 3 in., respectively.

Ex. 69. A tapering hollow iron column, 1 in. thick, is 24 ft. long, 10 in. in outside diameter at one end and 8 in. in diameter at the other. How many cubic inches of metal are there in it?

Ex. 70. Prove that a frustum of a circular cone is equal to three cones whose common altitude is the altitude of the frustum, and whose bases equal the lower base, the upper base, and the mean proportional between the bases of the frustum.

Ex. 71. The area of the entire surface of a frustum of a cone of revolution is 306π sq. in., and the radii of its bases are 11 in. and 5 in., respectively. Find the lateral area and the volume of it.

Ex. 72. The volume of a frustum of a right circular cone is 6020π cu. in., its altitude is 60 in., and the radius of its lower base is 15 in. Find the radius of the upper base and its lateral area.

Ex. 73. Find the diameter and the area of the surface of a sphere whose volume is $11\frac{2}{3}\pi$ cu. in.

Ex. 74. The altitude of a frustum of a cone of revolution is $3\frac{1}{2}$, and the radii of its bases are 5 and 3; what is the diameter of an equal sphere?

Ex. 75. Find the area of the surface and the volume of a sphere circumscribing a cylinder of revolution, the radius of whose base is 9, and whose altitude is 24.

Ex. 76. A cone of revolution is inscribed in a sphere whose diameter is $\frac{4}{3}$ the altitude of the cone. Prove that its lateral surface and volume are, respectively, $\frac{8}{3}$ and $\frac{9}{32}$ the surface and volume of the sphere.

Ex. 77. Given the area of the surface of a sphere S to find its volume.

Ex. 78. Given the volume of a sphere V to find the area of its surface.

Ex. 79. A portion of a plane bounded by an equilateral triangle, whose side is 6, revolves about one of its sides as an axis. Find the area of the entire surface, and the volume of the solid generated.

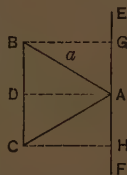
Ex. 80. A circular sector whose central angle is 45° and radius 12 revolves about a diameter perpendicular to one of its bounding radii. Find the volume of the spherical sector generated.

Ex. 81. A portion of a plane bounded by a right triangle, whose legs are a and b , revolves about its hypotenuse as an axis. Find the area of the entire surface, and the volume of the solid generated.

Ex. 82. A portion of a plane bounded by an equilateral triangle, whose altitude is h , revolves about one of its altitudes as an axis. Find the area of the surface, and the volume of the solid generated.

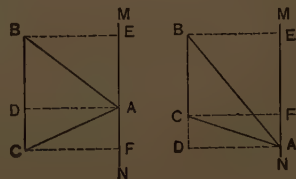
Ex. 83. A portion of a plane bounded by an equilateral triangle, whose side is a , revolves about a straight line drawn through one of its vertices parallel to the opposite side. Find the area of the entire surface, and the volume of the solid generated.

(The solid generated is the difference of the cylinder generated by $BCHG$, and the cones generated by ABG and ACH .)



Ex. 84. If a portion of a plane bounded by any triangle be revolved about an axis in its plane, not parallel to its base, which passes through its vertex without intersecting its surface, the volume of the solid generated is equal to the area of the surface generated by the base, multiplied by one third the altitude.

Ex. 85. If a portion of a plane bounded by any triangle be revolved about an axis which passes through its vertex parallel to its base, the volume of the solid generated is equal to the area of the surface generated by the base, multiplied by one third the altitude.



Ex. 86. Find the volume of a spherical sector, the altitude of whose base is 12, the diameter of the sphere being 25.

IMPORTANT DEFINITIONS AND THEOREMS OF PLANE GEOMETRY

§ 8. (a) One and only one straight line can be drawn through two points.

(b) A straight line can be extended indefinitely in each direction.

§ 11. Two straight lines can intersect at only one point.

§ 14. The straight line segment is the shortest line between two points.

§ 16. A circle is a closed curved line (in a plane) all points of which are equidistant from a point within called the center.

§ 17. All radii of the same circle or of equal circles are equal.

§ 20. An angle is the figure formed by two rays drawn from the same point.

§ 24. Adjacent angles are two angles that have a common vertex and a common side between them.

§ 26. If one straight line meets another straight line so that the adjacent angles formed are equal, each of these angles is a right angle.

§ 27. All right angles are equal.

§ 29. Two lines are perpendicular if they form a right angle.

§ 34. The sum of all the successive adjacent angles around a point on one side of a straight line is one straight angle.

§ 35. The sum of all the successive adjacent angles around a point is two straight angles.

§ 36. Two angles are complementary if their sum equals a right angle.

§ 37. Complements of the same angle or of equal angles are equal.

§ 38. Two angles are supplementary if their sum is equal to a straight angle.

§ 39. If two adjacent angles have their exterior sides in a straight line, they are supplementary.

§ 40. If two adjacent angles are supplementary, their exterior sides are in a straight line.

§ 41. Supplements of the same angle or of equal angles are equal.

§ 51. Ax. 1. Quantities which are equal to the same quantity or to equal quantities are equal to each other.

Ax. 2. Any quantity may be substituted for its equal in a mathematical expression.

Ax. 3. If equals be added to equals, their sums are equal.

Ax. 4. If equals be subtracted from equals, their remainders are equal.

Ax. 5. If equals be multiplied by equals, the products are equal.

Ax. 6. If equals be divided by equals, the quotients are equal.

Ax. 7. The whole is equal to the sum of its parts.

Ax. 8. The whole is greater than any of its parts.

Ax. 9. If a and b are any two magnitudes of the same kind, then a is less than b , is equal to b , or is greater than b .

§ 60. Two geometrical figures are congruent if they can be made to coincide.

§ 61. Ax. 15. Two figures which are congruent to the same figure are congruent to each other.

§ 62. A geometrical figure may be moved about in space without changing any of its parts.

§ 63. If two triangles have two sides and the included angle of one equal respectively to two sides and the included angle of the other, the triangles are congruent.

§ 67. If two triangles have two angles and the included side of one equal respectively to two angles and the included side of the other, the triangles are congruent.

§ 69. In an isosceles triangle, the angles opposite the equal sides are equal.

§ 70. An equilateral triangle is also equiangular.

§ 73. If two triangles have the three sides of one equal respectively to the three sides of the other, the triangles are congruent.

§ 77. If two points are each equidistant from the ends of a segment, they determine the perpendicular-bisector of the segment.

§ 81. One and only one perpendicular can be drawn to a line at a point of the line.

§ 83. One and only one perpendicular can be drawn to a line from a point outside the line.

§ 87. An exterior angle of a triangle is greater than either remote interior angle.

§ 89. Two straight lines are parallel if they lie in the same plane and do not meet however far they are extended.

§ 90. Ax. 16. Through a given point there can be only one parallel to a given line.

§ 91. If two lines are parallel to a third line, they are parallel to each other.

§ 93. If two lines are cut by a transversal so that a pair of alternate interior angles are equal, the lines are parallel.

§ 96. If two lines are cut by a transversal so that a pair of corresponding angles are equal, the lines are parallel.

§ 97. If two lines are perpendicular to a third line, they are parallel.

§ 98. If two lines are cut by a transversal so that a pair of interior angles on the same side of the transversal are supplementary, the lines are parallel.

§ 100. If two parallel lines are cut by a transversal, alternate interior angles are equal.

§ 101. If two parallels are cut by a transversal, corresponding angles are equal.

§ 102. If a line is perpendicular to one of two parallels, it is perpendicular to the other also.

§ 103. If two parallels are cut by a transversal, interior angles on the same side of the transversal are supplementary.

§ 105. If two angles have their sides respectively parallel, they are equal, provided both pairs of parallels extend in the same directions from their vertices, or in opposite directions.

§ 106. The sum of the angles of any triangle is a straight angle.

§ 108. A triangle cannot have two right angles or two obtuse angles.

§ 109. The acute angles of a right triangle are complementary.

§ 110. An exterior angle of a triangle equals the sum of the two remote interior angles.

§ 111. If two angles of one triangle are equal respectively to two angles of another triangle, the third angles are equal.

§ 112. If two triangles have a side, the opposite angle, and another angle of one equal respectively to a side, the opposite angle, and another angle of the other, the triangles are congruent.

§ 113. If two angles have their sides respectively perpendicular, they are either equal or supplementary.

§ 114. If two right triangles have the hypotenuse and a leg of one

equal respectively to the hypotenuse and a leg of the other, the triangles are congruent.

§ 115. If two equal obliques are drawn to a line from a point in a perpendicular to the line:

- (1) they cut off equal distances from the foot of the perpendicular;
- (2) they make equal angles with the perpendicular;
- (3) they make equal angles with the given line.

§ 118. I. Any point in the perpendicular-bisector of a segment is equidistant from the ends of the segment.

II. Any point equidistant from the ends of a segment lies in the perpendicular-bisector of the segment.

§ 119. Two obliques, drawn to a line from a point in a perpendicular to the line and cutting off equal segments from the foot of the perpendicular, are equal.

§ 120. I. Any point in the bisector of an angle is equidistant from the sides of the angle.

II. Any point equidistant from the sides of an angle lies in the bisector of the angle.

§ 121. Any point not in the bisector of an angle is unequally distant from the sides of the angle.

§ 123. If two angles of a triangle are equal, the sides opposite are equal, and the triangle is isosceles.

§ 124. If a triangle is equiangular, it is also equilateral.

§ 125. A polygon is a closed broken line (in a plane).

§ 126. A polygon is convex if no side, when extended, will pass through the interior of the polygon.

§ 131. A parallelogram is a quadrilateral whose opposite sides are parallel.

§ 132. Two consecutive angles of a parallelogram are supplementary.

§ 133. A diagonal of a parallelogram divides it into two congruent triangles.

§ 134. The opposite sides of a parallelogram are equal.

§ 135. The opposite angles of a parallelogram are equal.

§ 136. Segments of parallels included between parallels are equal.

§ 137. The diagonals of a parallelogram bisect each other.

§ 138. If two sides of a quadrilateral are equal and parallel, the figure is a parallelogram.

§ 139. If the opposite sides of a quadrilateral are equal, the figure is a parallelogram.

§ 147. If three or more parallels intercept equal lengths on one transversal, they intercept equal lengths on all transversals.

§ 148. If a line bisects one side of a triangle, and is parallel to a second side, it bisects the third side also.

§ 149. If a line is parallel to the bases of a trapezoid and bisects one of the non-parallel sides, it bisects the other also.

§ 151. If a line joins the mid-points of two sides of a triangle, it is parallel to the third side and equal to one half of it.

§ 153. The median of a trapezoid is parallel to the bases and equal to one half their sum.

§ 154. The sum of the interior angles of a polygon having n sides is $(n - 2)$ straight angles.

§ 155. If the sides of any polygon be extended in order to form an exterior angle at each vertex, the sum of these exterior angles is two straight angles.

§ 158. Axioms for combining Inequalities.

Ax. 17. If equals be added to unequals, the sums are unequal in the same order.

Ax. 18. If equals be subtracted from unequals, the differences are unequal in the same order.

Ax. 19. If unequals be added to unequals in the same order, the sums are unequal in the same order.

Ax. 20. If unequals be subtracted from equals or from unequals of opposite order, the differences are unequal and of order opposite to that of that subtrahend.

Ax. 21. If $a > b$ and $b > c$, then $a > c$.

§ 159. Fundamental Inequalities for Segments.

(a) Any side of a triangle is less than the sum of the other two sides.

(b) Any side of a triangle is greater than the difference between the other two sides.

§ 161. If two sides of a triangle are unequal, the angles opposite are unequal, the angle opposite the greater side being the greater.

§ 162. If two angles of a triangle are unequal, the sides opposite are unequal, the side opposite the greater angle being the greater.

§ 163. The hypotenuse of a right triangle is greater than either leg of the triangle.

§ 164. The perpendicular from a point to a line is the shortest segment from the point to the line.

§ 165. If two oblique segments, drawn from a point in a perpendicular to a line, cut off unequal distances from the foot of the perpendicular, the more remote is the greater.

§ 166. If two triangles have two sides of one equal respectively to two sides of the other, but the included angle of the first greater than the included angle of the second, then the third side of the first is greater than the third side of the second.

§ 167. If two triangles have two sides of one equal respectively to two sides of the other, but the third side of the first greater than the third side of the second, then the angle opposite the third side of the first is greater than the angle opposite the third side of the second.

§ 169. The bisectors of the interior angles of a triangle meet at a point which is equidistant from the sides of the triangle.

§ 170. The perpendicular-bisectors of the sides of a triangle meet at a point which is equidistant from the vertices of the triangle.

§ 171. The altitudes of a triangle meet at a point.

§ 172. The medians of a triangle meet at a point which lies two thirds the distance from each vertex to the mid-point of the opposite side.

§ 175. (a) If a straight line cuts a circle, it intersects it in two and only two points.

(b) If two circles intersect, they have two and only two points of intersection.

§ 181. In the same circle or in equal circles, if central angles are equal, they intercept equal arcs.

§ 182. In the same circle or in equal circles, if arcs are equal, the central angles which intercept them are equal.

§ 183. It may be proved that : in the same circle or in equal circles,
(a) The greater of two unequal central angles intercepts the greater arc ;

(b) The greater of two unequal arcs is intercepted by the greater central angle.

§ 185. In the same circle or in equal circles, if chords are equal, they subtend equal arcs.

§ 186. In the same circle or in equal circles, if arcs are equal, the chords which subtend them are equal.

§ 187. In the same circle or in equal circles, if two minor arcs are

unequal, then their chords are unequal, the greater arc being subtended by the greater chord.

§ 188. In the same circle or in equal circles, if two chords are unequal, then they subtend unequal minor arcs, the greater chord subtending the greater arc.

§ 189. If a diameter is perpendicular to a chord, it bisects the chord and its subtended arcs.

§ 190. A line through the center of a circle perpendicular to a chord bisects the chord.

§ 191. The perpendicular-bisector of a chord passes through the center of the circle, and bisects the arcs subtended by the chord.

§ 192. In the same circle or in equal circles, if chords are equal, they are equidistant from the center.

§ 193. In the same circle or in equal circles, if chords are equidistant from the center, they are equal.

§ 195. In the same circle or in equal circles, the less of two unequal chords is at the greater distance from the center of the circle.

§ 196. In the same circle or in equal circles, if two chords are unequally distant from the center, the more remote is the smaller.

§ 198. A straight line perpendicular to a radius at its outer extremity is a tangent to the circle.

§ 199. A tangent to a circle is perpendicular to the radius drawn to the point of contact.

§ 200. A line perpendicular to a tangent at its point of contact passes through the center of the circle.

§ 201. A line from the center of a circle perpendicular to a tangent passes through the point of contact.

§ 202. The tangents to a circle from an outside point are equal.

§ 205. If two circles are tangent to each other, their line of centers passes through the point of contact.

§ 206. If the distance between the centers of two circles equals the sum of their radii, the circles are tangent externally.

§ 207. If two circles intersect, the straight line joining their centers bisects their common chord at right angles.

§ 208. Parallel lines intercept equal arcs on a circle.

§ 213. In the same circle or in equal circles, two central angles have the same ratio as their intercepted arcs.

§ 217. An inscribed angle is measured by one half its intercepted arc

§ 220. The angle formed by a tangent and a chord drawn to the point of contact is measured by one half its intercepted arc.

§ 221. The angle formed by two chords intersecting within a circle is measured by one half the sum of the arcs intercepted by it and its vertical angle.

§ 222. The angle formed by two secants intersecting outside the circle is measured by one half the difference between its intercepted arcs.

§ 223. The angle formed by a secant and a tangent is measured by one half the difference between its intercepted arcs.

§ 224. The angle formed by two tangents is measured by one half the difference between its intercepted arcs.

§ 249. The mean proportional between two numbers is the square root of their product.

§ 250. In a proportion, the product of the extremes is equal to the product of the means.

§ 251. If three terms of one proportion are equal respectively to the three corresponding terms of another proportion, the fourth terms also are equal.

§ 252. If the product of two numbers is equal to the product of two other numbers, one pair may be made the means of a proportion having the other pair as the extremes.

§ 253. In any proportion, the terms are in proportion by Alternation; that is, the first term is to the third as the second is to the fourth.

§ 254. In any proportion, the terms are in proportion by Inversion; that is, the second term is to the first as the fourth is to the third.

§ 255. In any proportion, the terms are in proportion by Composition; that is, the sum of the first two terms is to the second as the sum of the last two terms is to the fourth.

§ 256. In any proportion, the terms are in proportion by Division; that is, the first term minus the second is to the second as the third minus the fourth is to the fourth.

§ 257. In any proportion, the terms are in proportion by Composition and Division; that is, the first term plus the second is to the first term minus the second as the third term plus the fourth is to the third term minus the fourth.

§ 258. In any proportion, like powers or like roots of the terms are in proportion.

§ 261. A parallel to one side of a triangle, intersecting the other two sides, divides the other two sides proportionally.

§ 268. A line which divides two sides of a triangle proportionally is parallel to the third side.

§ 270. In any triangle, the bisector of an interior angle divides the opposite side internally into segments proportional to the adjacent sides of the triangle.

§ 273. Two triangles are similar if they are mutually equiangular.

§ 274. Two triangles are similar if two angles of one are equal respectively to two angles of the other.

§ 275. Two right triangles are similar if an acute angle of one is equal to an acute angle of the other.

§ 276. Two triangles are similar if their sides are parallel each to each.

§ 277. Two triangles are similar if their sides are perpendicular each to each.

§ 280. Two triangles are similar if an angle of one equals an angle of the other and the sides including these angles are proportional.

§ 281. Two triangles are similar if their homologous sides are proportional.

§ 282. Homologous altitudes of similar triangles have the same ratio as any two homologous sides.

§ 283. If two chords are drawn through a fixed point within a circle, the product of the segments of one is equal to the product of the segments of the other.

§ 286. If any two secants are drawn through a fixed point outside a circle, the product of one and its external segment equals the product of the other and its external segment.

§ 287. If a secant and a tangent are drawn to a circle from the same point outside the circle, the square of the tangent is equal to the product of the whole secant and its external segment.

§ 288. If the altitude be drawn to the hypotenuse of a right triangle :

I. The altitude is the mean proportional between the segments of the hypotenuse.

II. Each leg is the mean proportional between the whole hypotenuse and the adjacent segment.

§ 291. The square of the hypotenuse of a right triangle is equal to the sum of the squares of the legs.

§ 293. Two polygons are similar if they are composed of the same number of triangles, similar each to each, and similarly placed.

§ 295. Two similar polygons can be divided into the same number of triangles, similar each to each, and similarly placed.

§ 296. In a series of equal ratios, the sum of the antecedents is to the sum of the consequents as any antecedent is to its consequent.

§ 297. The perimeters of two similar polygons have the same ratio as any two homologous sides.

§ 304. Parallel lines intercept proportional segments on all transversals.

§ 306. In any triangle, the bisector of an exterior angle at any vertex divides the opposite side externally into two segments whose ratio equals the ratio of the two adjacent sides of the triangle.

§ 310. In any triangle, the square of the side opposite an acute angle is equal to the sum of the squares of the other two sides, minus twice the product of one of these sides and the projection of the other upon it.

§ 311. In any triangle having an obtuse angle, the square of the side opposite the obtuse angle is equal to the sum of the squares of the other two sides, plus twice the product of one of these sides and the projection of the other upon it.

§ 313. When the three sides of a triangle are a , b , and c , the altitude to side a is $\frac{2}{a} \sqrt{s(s-a)(s-b)(s-c)}$, where $s = \frac{1}{2}(a+b+c)$.

§ 314. In any triangle, the sum of the squares of two sides equals twice the square of half the third side plus twice the square of the median drawn to that side.

§ 315. The difference between the squares of two sides of a triangle equals twice the product of the third side and the projection of the median upon that side.

§ 316. In any triangle, the product of two sides equals the product of the diameter of the circumscribed circle and the altitude upon the third side.

§ 318. In any triangle, the product of any two sides is equal to the product of the segments of the third side formed by the bisector of the opposite angle, plus the square of the bisector.

§ 323. Two congruent figures are necessarily equal, but two equal figures are not necessarily congruent.

§ 327. Two rectangles having equal altitudes are to each other as their bases.

§ 328. Two rectangles having equal bases are to each other as their altitudes.

§ 329. Two rectangles are to each other as the products of their bases by their altitudes.

§ 330. The area of a rectangle is the product of its base and altitude.

§ 331. The area of a parallelogram equals the product of its base and altitude.

§ 332. (1) Parallelograms having equal bases and equal altitudes are equal.

(2) Two parallelograms are to each other as the products of their bases by their altitudes.

(3) Parallelograms having equal altitudes are to each other as their bases.

(4) Parallelograms having equal bases are to each other as their altitudes.

§ 333. The area of a triangle equals one half the product of its base and altitude.

§ 334. (1) Triangles having equal bases and equal altitudes are equal.

(2) Two triangles are to each other as the products of their bases by their altitudes.

(3) Triangles having equal altitudes are to each other as their bases.

(4) Triangles having equal bases are to each other as their altitudes.

(5) A triangle is one half a parallelogram having the same base and altitude.

§ 337. The area of a trapezoid equals one half its altitude multiplied by the sum of its bases.

§ 338. The area of a trapezoid equals the product of its altitude and its median.

§ 340. The square upon the hypotenuse of a right triangle is equal to the sum of the squares upon the two legs of the triangle.

§ 343. The areas of two similar triangles are to each other as the squares of any two homologous sides.

§ 344. The areas of two similar polygons are to each other as the squares of any two homologous sides.

§ 346. Two triangles having an angle of one equal to an angle of the other are to each other as the products of the sides including these angles.

§ 357. Each angle of a regular polygon having n sides is $\left(\frac{n-2}{n}\right)$ st. $\angle$.

§ 358. A circle can be circumscribed about any regular polygon.

§ 359. A circle can be inscribed in any regular polygon.

§ 361. The central angle of a regular n -gon is $\frac{360^\circ}{n}$.

§ 363. The area of a regular polygon is equal to one half the product of its apothem and its perimeter.

§ 364. If a circle be divided into any number of equal arcs, the

chords of these arcs form a regular inscribed polygon of that number of sides.

§ 365. If from the mid-point of each arc subtended by a side of a regular polygon lines be drawn to its extremities, a regular inscribed polygon of double the number of sides is formed.

§ 366. An equilateral polygon inscribed in a circle is regular.

§ 367. If a circle is divided into any number of equal arcs, the tangents at the points of division form a regular circumscribed polygon of that number of sides.

§ 368. Tangents drawn to the circle at the mid-points of the arcs included between two consecutive points of contact of a regular circumscribed polygon form, with the sides of the original circumscribed polygon, a regular circumscribed polygon having double the number of sides.

§ 369. Tangents drawn to the circle at the mid-points of the arcs subtended by the sides of a regular inscribed polygon form a regular circumscribed polygon of the same number of sides.

§ 385. Regular polygons of the same number of sides are similar.

§ 386. The perimeters of two regular polygons of the same number of sides have the same ratio as their radii, or as their apothems.

§ 387. The areas of two regular polygons of the same number of sides have the same ratio as the squares of their radii or as the squares of their apothems.

§ 390. The circumference of a circle equals $2\pi r$, where r equals the number of linear units in the radius.

§ 391. The circumference of two circles have the same ratio as their diameters or as their radii.

§ 392. The area of a circle is one half the product of its radius and its circumference.

§ 395. The areas of two circles have the same ratio as the squares of their radii or of their diameters.

§ 397. The area of a sector is one half the product of the radius and the length of the arc intercepted by its angle.

401. A Variable is a number which assumes different values during a particular discussion.

A Constant is a number which has a fixed value throughout a particular discussion.

A Limit of a Variable is a constant such that the numerical value of the difference between the constant and the variable becomes and remains less than any small positive number.

We say a variable approaches its limit.

" $\doteq$ " is the symbol used for the words "approaches the limit."

§ 402. Ax. 22. If an increasing variable is always less than some constant, then it approaches a limit which is less than or equal to that constant. If a decreasing variable is always greater than some constant, then it approaches a limit which is greater than or equal to that constant.

§ 403. (a) If a variable x approaches a finite limit l , then cx , where c is a constant, approaches the limit cl .

(b) If two variables are constantly equal and each approaches a finite limit, then their limits are equal.

§ 405. In a sequence of regular inscribed polygons, the length of the side of the polygon is a decreasing variable which approaches zero as limit, as the number of sides increases indefinitely.

§ 406. In a sequence of regular inscribed polygons, the length of the apothem is an increasing variable which approaches the radius of the circle as limit.

§ 412. The length of a circle is the limit of the perimeter of any regular inscribed polygon as the number of sides is indefinitely increased.

§ 413. The perimeter of any regular circumscribed polygon approaches the circumference of the circle as limit if the number of sides is indefinitely increased.

§ 417. The area of a circle is the limit of the area of a regular inscribed polygon as the number of sides increases indefinitely.

§ 418. The area of any regular circumscribed polygon approaches the area of the circle as a limit as the number of sides is increased indefinitely.

§ 419. The area of a circle is the product of one half its radius and its Circumference.

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